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High-order Absorbing Boundary Conditions for anisotropic and convective wave equations

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ABSTRACT

High-order Absorbing Boundary Conditions (ABCs), applied on a rectangular artificial computational boundary that truncates an unbounded domain, are constructed for a general two-dimensional linear scalar time-dependent wave equation which represents acoustic wave propagation in *anisotropic* and subsonically *convective media*. They are extensions of the construction of Hagstrom, Givoli and Warburton for the isotropic stationary case. These ABCs are local, and involve only low-order derivatives owing to the use of auxiliary variables on the artificial boundary. The accuracy and well-posedness of these ABCs is analyzed. Special attention is given to the issue of mismatch between the directions of phase and group velocities, which is a potential source of concern. Numerical examples for the anisotropic case are presented, using a finite element scheme.

1. Introduction

The need for artificial computational boundaries in the solution of exterior wave problems, called "absorbing boundaries" among other names, arises quite often in various fields of application; acoustics, solid-earth geophysics and oceanography are important examples. It is remarkable that after more than three decades of research on the subject there are still a plethora of unresolved questions. Nonetheless, for isotropic problems in uniform media, methods which combine guaranteed accuracy, low cost, and geometric flexibility are now available. Here we take the first steps to extend these methods to more complex situations.

Since the mid 1990s two classes of methods have emerged as especially powerful; see, e.g., the review articles [1] and [2]. The first one is the Perfectly Matched Layer (PML) method, devised by Bérenger [3] in 1994 and since then further developed, analyzed and used by many authors. The second method is that of using high-order Absorbing Boundary Conditions (ABCs), which are local and involve no high derivatives. The first such ABC was devised by Collino [4] in 1993, and a few other formulations followed by other authors.

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Although usually derived by very distinct analyses, recent work has shown that, on the discrete level, the two methods are in fact quite closely related. In particular, it is shown in [5] how to design a nonstandard PML with a purely imaginary mesh continuation to exactly annihilate propagating waves at any incidence angle. This nonstandard PML is formally equivalent to the high-order ABCs proposed by Hagstrom and Warburton [6], which will be the approach taken here. The connection with PML will be further discussed when the proposed conditions take concrete form.

The use of ABCs has been very popular since the early 1970s – see the survey in [7] – but the development pioneered by Collino is the ability to implement ABCs of an *arbitrarily high-order*. In theory, some of the classical ABCs, such as the Engquist–Majda ABCs [8], the Bayliss–Turkel ABCs [9] and the Higdon ABCs [10,11], can be defined up to any desired order. However, the appearance of increasingly high-order derivatives in these ABCs renders them impractical beyond a certain order, typically 2 or 3. For example, the P-order Higdon ABC involves P-order derivatives in space and time, and is thus very inconvenient for implementation when P is large.

In contrast, the high-order ABCs devised by Collino [4], Grote and Keller [12,13], Hagstrom and Hariharan [14] Guddati and Tassoulas [15], Givoli and Neta [16,17] and Hagstrom and Warburton [6] involve no high derivatives owing to the use of special auxiliary variables $\phi_j (j = 1, \ldots, P)$ on the artificial boundary. The schemes are implemented for any order P, which is simply an input parameter provided by the user. Moreover, the computational cost increases only linearly with P. See the review [18].

It is worth mentioning that the present ABC differs from the Collino ABC in several ways. First, it is derived in a totally different way which seems more amenable for generalizations. Second, it is more general in the same way that the original Higdon condition is more general than the Engquist–Majda condition. Namely, our ABC involves free parameters, denoted here a_j, σ_j. In the isotropic case the parameters are related respectively to angles of incidence of propagating waves and decay rates of evanescent modes that are to be exactly annihilated. The Collino ABC is a "diagonalized" version of the proposed ABC for the simplest choice $a_j = 1, \sigma_j = 0$ for all j (in the isotropic case). The extra freedom entailed in our ABC allows one to use an adaptive scheme for choosing the best parameters dynamically during the solution process, as demonstrated in [19], or determine them a priori using quadrature rules as in [20]. (The investigation in [19] also shows, based on the Diaz–Joly theory [21], that for a non-adaptive scheme and short time computations the "Padé choice" $a_j \equiv 1$ is a very reasonable choice, which will be used in the illustrative computations presented here. More optimal parameters as used in [20] will be tested in subsequent work.) Third, the proposed ABC has the property that the norm of the auxiliary function ϕ_j is reduced with increasing j [22]. This property, which is not shared by the Collino ABC (nor by the Givoli–Neta ABC), contributes to the enhanced stability of the ABC for very large values of P, and allows the use of the norm of ϕ_P (the last ϕ_j) as an *a posteriori* error estimator, indicating whether the order P used is sufficiently large or needs to be increased [19].

The proposed ABC was originally devised for the standard scalar wave equation and for a first-order hyperbolic system [6]. Later it was improved and extended in several ways in [23]. The extensions included application to a dispersive medium, for which the Klein–Gordon wave equation governs, and to a stratified medium, either varying continuously or layered. Evanescent modes in the exact solution were also taken into account. See [20,24] for analysis and error estimates for the ABC based on complete expansion in propagating and decaying modes. The ABC was incorporated in both finite difference and finite element schemes.

All the high-order ABCs mentioned above, with the exception of [25], were developed for time-dependent waves in isotropic stationary media. In the present paper we extend our formulation to *anisotropic* and subsonically *convective* media. The convective wave equation plays a crucial role in aeroacoustic applications [26]. The subsonic background flow may be oriented in any direction with respect to the artificial boundary. Early important work on (low-order) ABCs for the convective wave equation was done by Rudy and Strikwerda [27] and Bayliss and Turkel [28,29]. Anisotropic media are very important in solid-earth geophysics [30]; although we consider here a scalar ("acoustic") equation and not the elastic equations (which are much more involved technically), the present investigation may serve as the first step in adapting the ideas underlying our ABC formulation to elastodynamics.

We analyze the accuracy and well-posedness of the new sequence of ABCs for general anisotropic and convective media in two dimensions. In doing so, we pay special attention to the issue of mismatch between the directions of phase and group velocities. Bécache et al. [31] showed that this mismatch plays an important role in the stability analysis of the PML. In fact, for some cases in anisotropic elasticity (and for convective acoustics without corrections) the PML becomes unstable due to this mismatch. See also [32] on PML stability and the recent paper [33] where a stabilized layer with a special damping profile is proposed (although without proof of still being perfectly matched). Here we show that the same stability issue arises in the analysis of the high-order ABC.

Following is the outline of the rest of the paper. In the next section we state the problem governed by a general two-dimensional scalar wave equation, from which the anisotropic wave equation and the convective wave equation may be obtained as special cases. We indicate the constraints assumed on the coefficients of the governing equation. In Section 3 we present the formulation of the high-order ABC for the general wave equation. This includes a discussion of the dispersion relation, the slowness curve and the phase-group speed mismatch. We first derive the basic (zero-order) ABC and then, based on it, the high-order ABC. In doing so we present its construction in a way slightly different and more illustrative than in previous publications.

In Section 4 we calculate the reflection coefficient, first for the basic ABC and then for the high-order ABC for both propagating and evanescent modes. In Section 5 we analyze the well-posedness of the new ABC. We do this in two ways. First we use the Kreiss-criterion for well-posedness. Second, we present some energy estimates by constructing energy functionals

which we show to be decreasing in time. For the anisotropic wave equation, we present a finite element formulation incorporating the new ABC in Section 6, and some numerical experiments in Section 7. We end with concluding remarks in Section 8.

2. Statement of the problem

We consider the two-dimensional time-dependent wave equation

$$\mathcal{L}u \equiv (\alpha_{11}\partial_x^2 + 2\alpha_{12}\partial_{xy} + \alpha_{22}\partial_y^2 - \partial_t^2 - 2\beta_1\partial_{tx} - 2\beta_2\partial_{ty} - \nu)u = 0, \tag{1}$$

in an unbounded domain $\mathcal{R}$, for the unknown field $u(x,y,t)$. Here and elsewhere $\partial_x, \partial_x^2 \equiv \partial_{xx}$, etc., denote partial derivatives with respect to the variables indicated. We assume that all the coefficients α_{ij}, β_i and ν are constant. It is assumed from the outset that

$$\alpha_{11}\alpha_{22} - \alpha_{12}^2 > 0, \tag{2}$$

so that in the steady state the equation is elliptic.

Physically, the wave Eq. (1) originates from the following one:

$$(\kappa_{11}\partial_x^2 + 2\kappa_{12}\partial_{xy} + \kappa_{22}\partial_y^2 - (\partial_t + V_1\partial_x + V_2\partial_y)^2 - \nu)u = 0. \tag{3}$$

In (3) we can identify the medium properties. The $\boldsymbol{\kappa} = [\kappa_{ij}]$ is the anisotropy tensor of wave-speed squares, the V_i are the background flow speed components, and ν is the dispersion parameter (which is nonzero, for example, when the whole system rotates). Thus, Eq. (3) represents anisotropic, convective and dispersive wave propagation. By comparing (3) with (1) we have,

$$\alpha_{11} = \kappa_{11} - V_1^2, \quad \alpha_{12} = \kappa_{12} - V_1V_2, \quad \alpha_{22} = \kappa_{22} - V_2^2, \tag{4}$$

$$\beta_1 = V_1, \quad \beta_2 = V_2. \tag{5}$$

We assume some constraints on the parameters, in addition to (2), from physical reasons. First, the wave-speed square tensor $\boldsymbol{\kappa}$ must be positive definite, i.e.,

$$\kappa_{11} > 0, \quad \kappa_{22} > 0, \quad \kappa_{11}\kappa_{22} - \kappa_{12}^2 > 0.$$

Second, we assume that the background flow is subsonic, namely $V < \sqrt{\kappa_{\min}}$, where $V = \sqrt{V_1^2 + V_2^2}$ and $\kappa_{\min}$ is the minimal principal value of $\boldsymbol{\kappa}$. This amounts to requiring

$$\beta_1^2 < \kappa_{11}, \quad \beta_2^2 < \kappa_{22},$$

and by using (4) and (5) we get simply

$$\alpha_{11} > 0, \quad \alpha_{22} > 0. \tag{6}$$

The conditions (2) and (6) together imply that the tensor $\boldsymbol{\alpha} = [\alpha_{ij}]$ is positive definite. Finally, we assume $\nu \geqslant 0$, so that the term νu in (1) indeed represents physical wave dispersion.

Eq. (1) reduces to simpler wave equations in specific cases. In case of a stationary non-dispersive medium ($\beta_i = 0,\ \nu = 0$), (1) leads to the *anisotropic wave equation*

$$\alpha_{11}\partial_x^2 u + 2\alpha_{12}\partial_{xy}u + \alpha_{22}\partial_y^2 u = \partial_t^2 u. \tag{7}$$

For an isotropic non-dispersive medium, (1) leads to the *convective wave equation*

$$(c^2 - \beta_1^2)\partial_x^2 u - 2\beta_1\beta_2\partial_{xy}u + (c^2 - \beta_2^2)\partial_y^2 u = \partial_t^2 u + 2\beta_1\partial_{tx}u + 2\beta_2\partial_{ty}u, \tag{8}$$

where c is the wave speed, introduced by taking $\kappa_{ij} = c^2\delta_{ij}$ (δ_{ij} being Kronecker's delta). Finally, for an isotropic stationary medium we get the Klein–Gordon (or dispersive) wave equation

$$c^2\nabla^2 u = \partial_t^2 u + \nu u. \tag{9}$$

If in addition the medium is non-dispersive ($\nu = 0$), (9) reduces to the classical wave equation.

The wave Eq. (1) is appended by the initial conditions

$$u(x,y,0) = u_0(x,y), \quad \partial_t u(x,y,0) = v_0(x,y). \tag{10}$$

We assume that the support of the initial functions u_0 and v_0 is compact. If the domain $\mathcal{R}$ is bounded by a physical boundary (say in the case of a semi-infinite domain or a wave-guide or a domain including a rigid scatterer) we also have to specify boundary conditions on this boundary; to simplify the presentation we assume that no such physical boundary is present.

Now we truncate the unbounded domain $\mathcal{R}$ by introducing the artificial boundary Γ. This divides the original domain into two subdomains: the computational domain Ω and the exterior domain D. We assume that Ω entirely encloses the support of u_0 and v_0. Thus,

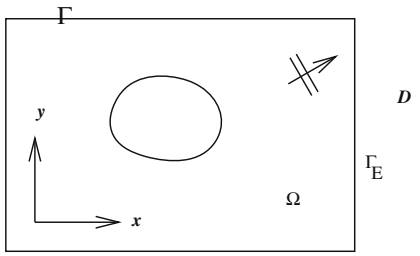

Fig. 1. Setup for an exterior problem. Shown are the computational artificial boundary Γ, the computational domain Ω and the exterior domain D.

$$u(x,y,0)=0, \quad \partial_t u(x,y,0)=0 \quad \text{in} \quad D\cup\Gamma.$$

The setup is shown in Fig. 1. In the next sections, we shall mostly consider Γ_E, the "east" side of Γ (namely the right edge of Ω in Fig. 1) whose normal and tangential directions are x and y, respectively. Analogous treatment can be applied to the other three sides.

The problem in Ω consists of the wave Eq. (1), the initial conditions (10) and a boundary condition – an ABC – on Γ. Our goal is to construct a high-order ABC that leads to a well-posed problem, is highly accurate, and can be practically implemented to any order. The accuracy requirement in the present context means that the solution (on the continuous level) of the new problem in Ω, using the ABC on Γ, is to be arbitrarily close to the restriction in Ω of the solution of the original problem in $\mathcal{R}$, for a sufficiently high ABC order P.

3. Formulation of the ABC

3.1. Dispersion relation, group velocity and slowness curve

For definiteness, in the discussion that follows we relate to the "east" side Γ_E of the artificial boundary Γ whose outward normal direction is x and tangent direction is y (see Fig. 1).

By definition, any ABC (or PML for that matter) has to be (approximately) transparent for *outgoing* waves and not admit incoming waves. A crucial point is that the words outgoing and incoming relate to the sign of the *group velocity* which carries the energy of the waves. In other words, the ABC must be satisfied by all waves whose x-component group velocity, V_{gx}, is positive, and only by these waves. This requires some caution since traditionally the absorbing operators are designed according to the *phase velocity* V_{px} and not according to V_{gx}. Given a plane-wave with angular frequency ω and wave vector $\boldsymbol{k}=\{k_1,k_2\}=\{k_x,k_y\}$, the two quantities V_{px} and V_{gx} are defined by

$$V_{px}=\frac{\omega}{k_1}, \quad V_{gx}=\frac{\partial\omega(k_1,k_2)}{\partial k_1}. \tag{11}$$

As long as V_{gx} and V_{px} have the same sign, as in the case for the standard isotropic wave equation (where in fact $V_{px}=V_{gx}=c$), no difficulty occurs. However, some anisotropic and convective media support waves associated with V_{gx} and V_{px} with opposite signs. In those cases the mismatch in sign may potentially render the ABC unstable.

In particular, consider a wave whose group velocity points inwardly (i.e., $V_{gx}<0$) but whose phase velocity points outwardly (i.e., $V_{px}>0$). Such a wave is incoming, due to the direction of its group velocity, and hence should *not* satisfy the ABC. However, if the ABC is designed to be satisfied by all waves with positive V_{px}, this wave *would* satisfy the ABC. In other words, the ABC would wrongly identify the wave as outgoing while it is in fact incoming, and this would lead to instability. However, by designing the ABC to allow only group-outgoing waves, one can avoid such instability. This is possible to do for the general wave Eq. (1) as we show here.

We start by deriving the dispersion relation associated with (1). By applying the Laplace transform in time and the Fourier transform in space to (1), or alternatively by substituting the plane-wave

$$u=\exp[i(\omega t-k_1x-k_2y)], \tag{12}$$

in (1), we find

$$\alpha_{11}k_1^2+(2\alpha_{12}k_2+2\beta_1\omega)k_1+\alpha_{22}k_2^2-\omega^2+2\beta_2\omega k_2+v=0. \tag{13}$$

This can be regarded as a quadratic equation for the x-wave-number k_1 given the angular frequency ω and the y-wave-number k_2. Thus we get two roots for k_1:

$$k_1=\frac{1}{\alpha_{11}}\left[-\alpha_{12}k_2-\beta_1\omega\pm\sqrt{(\alpha_{12}k_2+\beta_1\omega)^2-\alpha_{11}(\alpha_{22}k_2^2-\omega^2+2\beta_2\omega k_2+v)}\right]. \tag{14}$$

Note that these roots can be real, corresponding to propagating modes, or complex, corresponding to evanescent modes.

For the propagating modes, the first question that we should ask is which of the two roots corresponds to an outgoing wave in the group velocity sense, i.e., to a wave with $V_{gx} > 0$. As promoted by Bécache et al. [31], a good way to visualize this issue is by means of the *slowness diagram*. For a given plane-wave, the slowness K is the inverse of the phase velocity, namely $K = 1/V_p = k/\omega$, where $k = (k_1^2 + k_2^2)^{1/2}$ is the wave-number. Accordingly, the slowness vector $\boldsymbol{K}$ is defined by

$$\boldsymbol{K} = \frac{\boldsymbol{k}}{\omega}. \tag{15}$$

The dispersion relation (13) implies a relation between ω and $\boldsymbol{K} = (K_1,\ K_2) = (K_x,\ K_y)$. The slowness diagram is obtained by drawing, for a fixed ω (say $\omega = 1$), the curve formed by the dispersion relation on the $(K_x,\ K_y)$ plane. In this diagram, one can easily draw the directions of both the phase velocity and the group velocity for each point $Q = (K_x,\ K_y)$ on the slowness curve; the phase velocity direction is obtained by connecting the origin with the point Q, whereas the group velocity direction is determined by the outward normal vector at Q. See [31] for more details.

For an isotropic medium, the slowness curve is a circle, and the directions of the phase and group velocities coincide. For an anisotropic medium they may differ. Fig. 2 shows a typical slowness diagram for the anisotropic wave Eq. (7). In Fig. 2(a) we consider two points on the curve in the first quadrant. Both points correspond to positive x-phase speeds. However, one point (marked by V_{gx}^+) corresponds to a positive x-group speed, while the other (marked by V_{gx}^-) corresponds to a negative x-group speed. Thus, the latter corresponds to a wave with a phase-group mismatch, as discussed above. A different perspective is shown in Fig. 2(b) for the same slowness curve. For a fixed K_y the dispersion relation yields two roots for K_x: one corresponds to a positive x-group speed (marked by K_x^+) and the other (marked by K_x^-) corresponds to a negative x-group speed.

Fig. 3 shows a similar diagram for the convective wave Eq. (8).

A conclusion from Figs. 2 and 3 is that of the two roots of k_1 in (14), *the larger root* (i.e., the one corresponding to the plus sign) is always associated with a positive V_{gx}, namely with a group-outgoing wave. This provides us with an easy way to control group-outgoing waves via the expression for the phase velocity (or the slowness), namely it allows us to work with (14) and always pick the + sign in order to identify waves whose group velocity is outgoing. This 'happy coincidence' is, unfortunately, lost in elasticity [31]. We emphasize that were we to derive an ABC which was satisfied by phase-outgoing waves whose group velocity is incoming, this ABC would have been unstable.

Despite this convenient geometrical interpretation, we shall not make use of it in the derivation. Instead, we shall derive the basic ABC in the next section directly by calculating the group velocity V_{gx}. It is possible to show that the two approaches are completely equivalent and lead to exactly the same results.

We now identify those plane waves for which there is a phase-group velocity mismatch. We write

$$k_1 = k\cos\theta, \quad k_2 = k\sin\theta, \tag{16}$$

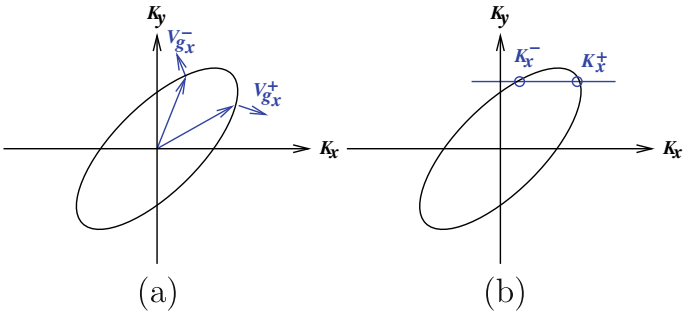

Fig. 2. Slowness diagram for an anisotropic medium: (a) two group velocity vectors; and (b) roots of K_x for a fixed K_y.

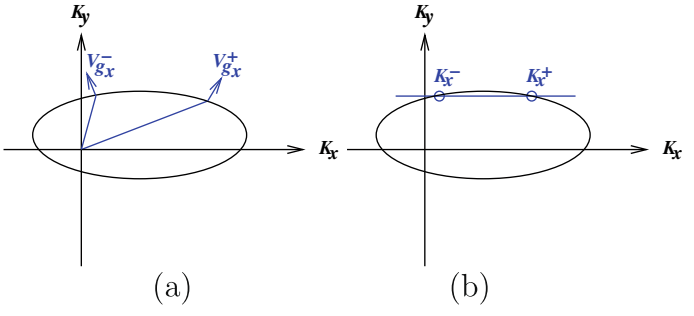

Fig. 3. Same as Fig. 2, but for a convective medium.

where θ is the angle of incidence. We consider waves which are "phase-outgoing" with respect to the east boundary, namely waves with $|\theta| < \pi/2$. Thus the phase velocity V_{px} is positive, and the condition for mismatch is $V_{gx} < 0$. By differentiating (13) with respect to k_1 and using $V_{gx} = \partial\omega/\partial k_1$ we obtain

$$2\alpha_{11}k_1 + 2\alpha_{12}k_2 + 2\beta_1\omega + 2\beta_1 k_1 V_{gx} - 2\omega V_{gx} + 2\beta_2 k_2 V_{gx} = 0. \tag{17}$$

By substituting (16) in (17) we find

$$V_{gx}(\theta) = \beta_1 + \frac{\beta_1^2 + \alpha_{11} + (\beta_1\beta_2 + \alpha_{12})\tan\theta}{\sqrt{I(\theta)}}, \tag{18}$$

where

$$I(\theta) = \beta_1^2 + \alpha_{11} + 2(\beta_1\beta_2 + \alpha_{12})\tan\theta + (\beta_2^2 + \alpha_{22})\tan^2\theta + \nu/(k^2\cos^2\theta). \tag{19}$$

From this we deduce that the condition for phase-group mismatch is

$$\beta_1\sqrt{I(\theta)} + \beta_1^2 + \alpha_{11} + (\beta_1\beta_2 + \alpha_{12})\tan\theta < 0. \tag{20}$$

For a stationary isotropic medium ($\beta_1 = \beta_2 = \alpha_{12} = 0, \alpha_{11} = \alpha_{22} = c^2$) the general condition (20) reduces to $\alpha_{11} = c^2 < 0$ which is impossible. Hence, in the isotropic case there is no phase-group mismatch. *For a stationary anisotropic medium* ($\beta_1 = \beta_2 = 0$) we find from (20) that the condition for mismatch is

$$\alpha_{11} + \alpha_{12}\tan\theta < 0. \tag{21}$$

If $\alpha_{12} = 0$ (orthotropy in the (x, y) directions) this never occurs. If $\alpha_{12} \neq 0$, this will hold if both $\alpha_{12}\tan\theta < 0$ and $\alpha_{11} < |\alpha_{12}\tan\theta|$. *For a convective isotropic non-dispersive medium*, (20) and (19) reduce to

$$c + \beta_1\sqrt{1 + \tan^2\theta} < 0. \tag{22}$$

If $\beta_1 > 0$, i.e., at an *outflow* boundary, this never holds, namely there is no phase-group mismatch. If $\beta_1 < 0$, i.e., at an *inflow* boundary, (22) yields

$$\tan^2\theta > \left(\frac{c}{\beta_1}\right)^2 - 1. \tag{23}$$

The right side is always positive, since in a subsonic regime $\beta_1 < c$. Making use of a trigonometric identity, the condition (23) reduces to

$$\cos\theta < \frac{|\beta_1|}{c} \equiv |M_1|, \tag{24}$$

where M_1 is the Mach number in the x direction.

We note that in the stationary case, the presence of wave dispersion ($\nu > 0$) does not affect the question of phase-group mismatch. On the other hand, in the convective case it does. If $\nu > 0$ then (24) is replaced by

$$\cos\theta < |M_1|\sqrt{1 + \nu/(ck)^2}.$$

We see that the presence of dispersion triggers phase-group mismatch for smaller angles θ; thus dispersion has a destabilizing effect in this sense.

Finally we consider the evanescent modes. Now it is simpler to rewrite (14) as

$$ik_1 = -i\frac{\alpha_{12}}{\alpha_{11}}k_2 - i\frac{\beta_1}{\alpha_{11}}\omega \pm \sigma, \quad \sigma > 0. \tag{25}$$

Using (25) in (12), it is clear that the outgoing solution is identified with the decaying solution corresponding to the "+" sign in (25).

3.2. Basic ABC

In this section we derive a zero-order ABCs for the general wave Eq. (1) that will serve as the basis for the high-order ABC to be developed later.

We first write the group velocity x-component V_{gx} in terms of the slowness components K_1 and K_2. From (15) we have

$$K_i = \frac{k_i}{\omega}. \tag{26}$$

Eqs. (17) and (26) lead to

$$V_{gx} = -\frac{\alpha_{11}K_1 + \alpha_{12}K_2 + \beta_1}{\beta_1 K_1 + \beta_2 K_2 - 1}.$$

We can easily invert this relation to obtain

$$K_1 = \frac{-A_1 + A_2 V_{gx}}{\alpha_{11} + \beta_1 V_{gx}}, \tag{27}$$

with

$$A_1 = \alpha_{12}K_2 + \beta_1, \quad A_2 = 1 - \beta_2 K_2.$$

Using (26) and (27), the quadratic Eq. (13) for k_1 can also be written as a quadratic equation for V_{gx}, i.e.,

$$c_2 V_{gx}^2 + 2c_1 V_{gx} + c_0 = 0, \tag{28}$$

where $c_2 = \alpha_{11}A_2^2 + 2\beta_1 A_1 A_2 + \beta_1^2 A_3$, $c_1 = -\beta_1 A_1^2 + \alpha_{11}\beta_1 A_3$, $c_0 = -\alpha_{11}A_1^2 + \alpha_{11}^2 A_3$, $A_3 = \alpha_{22}K_2^2 + 2\beta_2 K_2 - 1 + \frac{v}{\omega^2}$. The solution of (28) is

$$(V_{gx})^{\pm} = \frac{-c_1 \pm \sqrt{\Delta}}{c_2}, \quad \Delta = c_1^2 - c_0 c_2.$$

Since we are interested in outgoing waves, we pick the positive root:

$$(V_{gx})^{+} = \frac{-c_1 + \sqrt{\Delta}}{c_2}. \tag{29}$$

Note that the procedure above guarantees that the waves thus identified are outgoing "in the correct sense," namely that their *group speed* component in the direction normal to the boundary is positive.

Up to this point we have made no approximation. Now we approximate the square root in (29) by a Taylor expansion to first-order in K_2. To this end we assume that $|K_2| \ll 1$. This corresponds to the usual requirement that the basic ABC be accurate for waves with almost normal incidence. In addition we will assume that $v/\omega^2 = O(K_2^2)$. This assumption guarantees that (14) yields two real roots, which correspond to propagating waves.

Under these assumptions we get, after some algebra,

$$\sqrt{\Delta} = (\lambda_1 - 2\lambda_2 K_2)\sqrt{\lambda_1} + O(K_2^2), \tag{30}$$

where

$$\lambda_1 = \alpha_{11} + \beta_1^2 > 0, \quad \lambda_2 = \beta_2\alpha_{11} - \beta_1\alpha_{12}. \tag{31}$$

Substituting (30) in (29) and the result in (27) yields

$$K_1^+ = \frac{\zeta_3 + \zeta_4 K_2 + O(K_2^2)}{\zeta_1 + \zeta_2 K_2 + O(K_2^2)}, \tag{32}$$

where K_1^+ is the root of K_1 which corresponds to the positive group velocity V_{gx}^+, and

$$\zeta_1 = \lambda_1^{3/2}(\beta_1 + \lambda_1^{1/2}) > 0, \quad \zeta_2 = -2\lambda_2\lambda_1^{1/2}(\beta_1 + \lambda_1^{1/2}),$$
$$\zeta_3 = \lambda_1^{3/2}, \quad \zeta_4 = -\lambda_1^{1/2}(2\lambda_2 + \beta_2\lambda_1) - \lambda_1(\alpha_{12} + \beta_1\beta_2).$$

Eq. (32) can be further approximated by a Taylor expansion of the ratio, still consistently retaining only terms up to first-order in K_2, to yield

$$K_1^+ = \frac{1}{\zeta_1}\left[\zeta_3 + \left(\zeta_4 - \frac{\zeta_3\zeta_2}{\zeta_1}\right)K_2\right] = \frac{1}{\alpha_{11}}\left[\lambda_1^{1/2} - \beta_1 - \left(\alpha_{12} + \frac{\lambda_2}{\lambda_1^{1/2}}\right)K_2\right]. \tag{33}$$

Some algebra is involved in deriving the second equality in (33) from the first. Using (26) and the definitions (31), the last result gives the new linearized dispersion relation

$$\sqrt{\alpha_{11} + \beta_1^2}\left(k_1 + \frac{\alpha_{12}}{\alpha_{11}}k_2 + \frac{\beta_1}{\alpha_{11}}\omega\right) - \left(\left(1 + \frac{\beta_1^2}{\alpha_{11}}\right)\omega - \left(\beta_2 - \frac{\alpha_{12}\beta_1}{\alpha_{11}}\right)k_2\right) = 0,$$

where we have omitted the $^+$ notation from k_1 for clarity.

Finally we are in a position to transform the last result back into physical space ($k_1 \to i\partial_x, k_2 \to i\partial_y, \omega \to -i\partial_t$) and thus to obtain the basic ABC

$$B_0 u \equiv \left[\sqrt{\alpha_{11} + \beta_1^2}\,\partial_x + \left(\beta_2 - \frac{\alpha_{12}\beta_1}{\alpha_{11}} + \frac{\alpha_{12}}{\alpha_{11}}\sqrt{\alpha_{11} + \beta_1^2}\right)\partial_y + \left(1 + \frac{\beta_1^2}{\alpha_{11}} - \frac{\beta_1}{\alpha_{11}}\sqrt{\alpha_{11} + \beta_1^2}\right)\partial_t\right]u = 0 \quad \text{on } \Gamma_E. \tag{34}$$

Exactly the same result may be obtained by approximating the square root in the expression for k_1 in (14), and choosing the plus sign in that expression. The justification in doing this comes from our observation (see previous section) that of the two roots of k_1 in (14), the larger root is always associated with a positive V_{gx}.

The expression for the basic ABC (34) reduces to simpler expressions for some special cases. For the *anisotropic wave equation* (7), (34) reduces to

$$B_0 u \equiv \left[\sqrt{\alpha_{11}}\partial_x + \frac{\alpha_{12}}{\sqrt{\alpha_{11}}}\partial_y + \partial_t\right]u = 0 \quad \text{on } \Gamma_E.$$

For the *convective wave equation* (8), Eq. (34) reduces (after further simplification) to

$$B_0 u \equiv [(c+\beta_1)\partial_x + \beta_2\partial_y + \partial_t]u = 0 \quad \text{on } \Gamma_E.$$

For the *dispersive wave equation* (9), and for the classical wave equation ($v = 0$), (34) reduces to

$$B_0 u \equiv [c\partial_x + \partial_t]u = 0 \quad \text{on } \Gamma_E. \tag{35}$$

The latter ABC is the classical 0th-order boundary condition, often called the Sommerfeld-like ABC.

It can easily be shown that in the absence of dispersion ($v = 0$) the general ABC (34) is *exact* for a plane-wave with *normal incidence* ($K_2 = 0$). This is quite obvious even without performing the algebra, since the approximation we have made was associated only with K_2 and v. For angles of incidence close to normal this ABC generates a small reflection error, while for oblique incidence it may generate large spurious reflection. This is the motivation for seeking higher-order ABCs.

Higdon, who considered the Klein–Gordon Eq. (9) in [11], constructed ABCs which are exact for plane waves with given angles of incidence. This led to a basic ABC extending (35), i.e.,

$$B_0 u \equiv [c\partial_x + a_0\partial_t]u = 0 \quad \text{on } \Gamma_E. \tag{36}$$

Here $0 < a_0 \leqslant 1$ is a parameter which has to be chosen and which signifies the cosine of the incidence angle θ. If $a_0 = \cos\theta$, the ABC (36) is exact. Of course, the incorporation of a parameter $a_0 \neq 1$ in a zero-order ABC does not seem beneficial, since in realistic problems there are many waves impinging the boundary simultaneously with different angles of incidence, and moreover one does not know in advance what these angles are. However, the idea becomes meaningful for a high-order ABC that involves many such parameters, which may be chosen in different ways [19]. To keep the same level of generality, we incorporate a parameter a_0 in our general ABC (34), although in this case a_0 does not have as simple a physical meaning. To do this, we rewrite the argument of the square root in (14) and recall that $\alpha_{11}\alpha_{22} - \alpha_{12}^2 > 0$. Assuming a real root we find:

$$\sqrt{(\alpha_{12}k_2 + \beta_1\omega)^2 - \alpha_{11}(\alpha_{22}k_2^2 - \omega^2 + 2\beta_2\omega k_2 + v)} = \sqrt{\alpha_{11}+\beta_1^2}\sqrt{D_0^2 - D_1^2} \equiv \sqrt{\alpha_{11}+\beta_1^2}aD_0, \tag{37}$$

where $0 < a \leqslant 1$ and

$$D_0 = \omega - \frac{\alpha_{11}\beta_2 - \alpha_{12}\beta_1}{\alpha_{11}+\beta_1^2}k_2,$$

$$D_1^2 = \left((\alpha_{11}\alpha_{22} - \alpha_{12}^2)(\alpha_{11}+\beta_1^2) + (\alpha_{11}\beta_2 - \alpha_{12}\beta_1)^2\right)\frac{k_2^2}{(\alpha_{11}+\beta_1^2)^2} + \frac{\alpha_{11}v}{\alpha_{11}+\beta_1^2}.$$

Thus we may approximate the root by $\sqrt{\alpha_{11}+\beta_1^2}a_0D_0$. Keeping track of a_0 throughout the derivation, we obtain an extended expression for the ABC (34), i.e.,

$$B_0 u \equiv (\xi\partial_x + \eta_0\partial_y + \tau_0\partial_t)u = 0 \quad \text{on } \Gamma_E, \tag{38}$$

where

$$\xi = \sqrt{\lambda_1} = \sqrt{\alpha_{11}+\beta_1^2}, \tag{39}$$

$$\eta_0 = \frac{1}{\alpha_{11}}\left(a_0\lambda_2 + \alpha_{12}\sqrt{\lambda_1}\right) = a_0\left(\beta_2 - \frac{\alpha_{12}\beta_1}{\alpha_{11}}\right) + \frac{\alpha_{12}}{\alpha_{11}}\sqrt{\alpha_{11}+\beta_1^2}, \tag{40}$$

$$\tau_0 = \frac{1}{\alpha_{11}}\left(a_0\lambda_1 - \beta_1\sqrt{\lambda_1}\right) = a_0\left(1 + \frac{\beta_1^2}{\alpha_{11}}\right) - \frac{\beta_1}{\alpha_{11}}\sqrt{\alpha_{11}+\beta_1^2}. \tag{41}$$

Here λ_1 and λ_2 are defined by (31). This ABC reduces to (34) when $a_0 = 1$ and reduces to (36) for a stationary isotropic medium.

Similarly, we can design a basic operator which is exact for a particular evanescent mode. Using (25) we write:

$$ik_1 \approx -i\frac{\alpha_{12}}{\alpha_{11}}k_2 - i\frac{\beta_1}{\alpha_{11}}\omega + \frac{\sigma_0}{\sqrt{\alpha_{11}+\beta_1^2}}, \quad \sigma_0 > 0. \tag{42}$$

This yields:

$$\Sigma_0 u \equiv \left(\xi\partial_x + \xi\frac{\alpha_{12}}{\alpha_{11}}\partial_y - \xi\frac{\beta_1}{\alpha_{11}}\partial_t + \sigma_0\right)u = 0 \quad \text{on } \Gamma_E. \tag{43}$$

3.3. Auxiliary variables and recursive relations

In the remainder of Section 3 we shall construct the high-order ABC. The motivation for doing this is that with our construction, the reflection coefficient generated by waves impinging on Γ will decrease exponentially fast with the order of the ABC. This will be shown in Section 4.

The construction of the high-order ABCs of the type considered here involves two major steps. The first step is to devise a boundary operator which is a product of a desired number of operators of the basic forms (38) and (43). This is the original idea of Higdon [10,11], which leads to a "theoretical" high-order ABC. It is "theoretical" because it involves increasingly high spatial and temporal derivatives. The second step is to render this ABC practical by eliminating all the high derivatives. This is made possible by introducing a sequence of auxiliary variables ϕ_j which obey certain relations. This second step was proposed by Givoli and Neta [16,17] and Hagstrom and Warburton [6].

To motivate the construction method for the high-order ABC, we start by considering the *isotropic wave equation* (9) and recalling Higdon's ABC devised for it [10,11]:

$$\left[\prod_{j=0}^{P}(a_j\partial_t + c\partial_x)\right]u = 0 \quad \text{on } \Gamma_E. \tag{44}$$

Note that the Pth-order operator in (44) is a product of $P+1$ basic operators of the form B_0 in (36). This ABC involves normal and temporal derivatives of order $P+1$, and thus is impractical for large P. To get rid of the high derivatives, Givoli and Neta [16] rewrote (44) in the following manner:

$$(a_0\partial_t + c\partial_x)u = \phi_1 \quad \text{in } D_E, \tag{45}$$

$$(a_j\partial_t + c\partial_x)\phi_j = \phi_{j+1}, \quad j = 1,\dots,P \quad \text{in } D_E, \tag{46}$$

$$\phi_{P+1} = 0 \quad \text{on } \Gamma_E. \tag{47}$$

Here the ϕ_j are auxiliary variables, and (45)–(47) are the *recursive relations* among them. The domain D_E is the east side of the exterior domain, namely $D_E = \overline{D_{ES} \cup D_{EC} \cup D_{EN}}$, as shown in Fig. 4, and is understood to include the boundary Γ_E. It is important to note that the relations (45) and (46) are actually *definitions* of the auxiliary variables $\phi_1,\dots,\phi_{P+1}$ in the domain D_E. The condition (47) which holds only on the boundary Γ_E is the additional information which turns (45)–(47) into a boundary condition. The ϕ_j satisfy zero initial conditions, i.e.,

$$\phi_j(x,y,0) = 0, \quad \partial_t\phi_j(x,y,0) = 0, \quad j = 1,\dots,P \quad \text{in } D_E. \tag{48}$$

It is easy to see that (45)–(47) is equivalent to (44).

Before presenting the alternative construction used here, we observe that although Eqs. (45)–(47) do not involve high derivatives, they do involve the *normal derivatives* $\partial_x\phi_j$, and hence they do not constitute a practical ABC themselves. Since the ϕ_j are to be discretized on the boundary Γ_E alone, we may allow their temporal and tangential derivatives but not their normal derivative to appear in the ABC. Thus (45)–(47) have to be manipulated in order to get rid of the $\partial_x\phi_j$. We shall see how this is done later.

The Hagstrom–Warburton formulation for the isotropic wave equation [6] is based on recursive relations which constitute a modification to (45)–(47), i.e.,

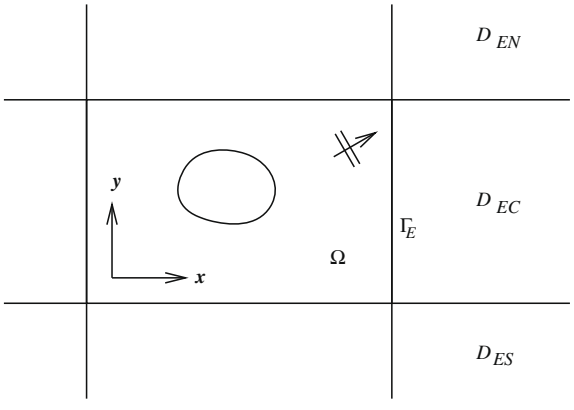

Fig. 4. Exterior domains related to the definition of the auxiliary variables ϕ_j.

$$(a_0\partial_t + c\partial_x)u = a_0\partial_t\phi_1 \quad \text{in } D_E, \tag{49}$$
$$(a_j\partial_t + c\partial_x)\phi_j = (a_j\partial_t - c\partial_x)\phi_{j+1}, \quad j = 1,\dots,P \quad \text{in } D_E, \tag{50}$$
$$\phi_{P+1} = 0 \quad \text{on } \Gamma_E. \tag{51}$$

A primary motivation for this modification is the fact that the auxiliary variables in (45)–(47) are obtained by successive differentiations and thus may suffer from an increasing loss of smoothness; here we counteract that possibility by imposing a recursion with balanced differentiations. In addition, we note that if the auxiliary variable indices are reinterpreted as grid indices the new recursion takes the form of a box scheme approximation to a nonstandard PML with "mesh width" $H_{j+1/2}^{-1} = \frac{a_j}{2}\partial_t$:

$$\partial_x\left(\frac{\phi_j + \phi_{j+1}}{2}\right) = \frac{\phi_{j+1} - \phi_j}{H_{j+1/2}}. \tag{52}$$

We now proceed to obtain a single condition from (49)–(51) in terms of u alone (i.e., a counterpart of (44)). To this end, we apply the operator $(a_1\partial_t + c\partial_x)$ to both sides of (49) and use (50) with $j = 1$ to obtain

$$(a_1\partial_t + c\partial_x)(a_0\partial_t + c\partial_x)u = a_0\partial_t(a_1\partial_t - c\partial_x)\phi_2. \tag{53}$$

Now we apply the operator $(a_2\partial_t + c\partial_x)$ to both sides of (53) and use (50) with $j = 2$. Continuing in this manner up to $j = P$ we obtain

$$\left[\prod_{j=0}^{P}(a_j\partial_t + c\partial_x)\right]u = a_0\partial_t\left[\prod_{j=1}^{P}(a_j\partial_t - c\partial_x)\right]\phi_{P+1}. \tag{54}$$

Note that we cannot deduce that the right side of (54) vanishes based on (51), since (51) does not tell us anything about $\partial_x\phi_{P+1}$ on Γ_E. Next we apply to both sides of (54) the operator appearing on the left side of (54) but starting the product from $j = 1$, to get

$$(a_0\partial_t + c\partial_x)\left[\prod_{j=1}^{P}(a_j\partial_t + c\partial_x)^2\right]u = a_0\partial_t\left[\prod_{j=1}^{P}\left(a_j^2\partial_t^2 - c^2\partial_x^2\right)\right]\phi_{P+1}. \tag{55}$$

Now we make use of a crucial fact, namely that ϕ_{P+1} satisfies the same wave Eq. (9) as u does. In fact, all the auxiliary variables ϕ_j satisfy this equation, i.e.,

$$[c^2(\partial_x^2 + \partial_y^2) - (\partial_t^2 + \nu)]\phi_j = 0, \quad j = 1,\dots,P+1 \quad \text{in } D_E. \tag{56}$$

We prove this in Appendix A (for the general wave Eq. (1)). Making use of (56), we can eliminate the ∂_x^2 appearing on the right side of (55) and write it in terms of temporal and tangential derivatives only. Then we can use (51) to deduce that the right side of (55) vanishes on Γ_E, namely

$$(a_0\partial_t + c\partial_x)\left[\prod_{j=1}^{P}(a_j\partial_t + c\partial_x)^2\right]u = 0 \quad \text{on } \Gamma_E. \tag{57}$$

This boundary condition should be contrasted with the original Higdon condition (44). The squaring of all the $j \geqslant 1$ terms in (57) leads to an ABC with higher accuracy.

To extend the recursive relations (49)–(51) to the case where the governing equation is the *general wave equation* (1), we note that the form of the extended relations can be deduced from the basic ABC (38) in the same manner that the form of (49)–(51) is deduced from (36). Including the evanescent mode operators (43) for completeness we find the extended relations to be

$$\mathcal{B}_0^+ u = \mathcal{B}_0^*\phi_1 \quad \text{in } D_E, \tag{58}$$
$$\mathcal{B}_j^+\phi_j = \mathcal{B}_j^-\phi_{j+1}, \quad j = 1,\dots,P \quad \text{in } D_E, \tag{59}$$
$$\Sigma_j^+\phi_{P+j} = \Sigma_j^-\phi_{P+j+1}, \quad j = 1,\dots,Q \quad \text{in } D_E, \tag{60}$$
$$\phi_{P+Q+1} = 0 \quad \text{on } \Gamma_E, \tag{61}$$

where the operators appearing in (58) and (59) are defined by

$$\mathcal{B}_j^+ \equiv \xi\partial_x + \eta_j^+\partial_y + \tau_j^+\partial_t, \quad j \geqslant 0, \tag{62}$$
$$\mathcal{B}_j^- \equiv -\xi\partial_x + \eta_j^-\partial_y + \tau_j^-\partial_t, \quad j \geqslant 0, \tag{63}$$
$$\mathcal{B}_0^* = \eta_0^*\partial_y + \tau_0^*\partial_t, \tag{64}$$
$$\Sigma_j^\pm \equiv \pm(\xi\partial_x + \eta\partial_y + \tau\partial_t) + \sigma_j, \quad j \geqslant 0, \tag{65}$$

with

$$\xi = \sqrt{\alpha_{11} + \beta_1^2}, \quad \eta = \frac{\alpha_{12}}{\alpha_{11}}\xi, \quad \tau = -\frac{\beta_1}{\alpha_{11}}\xi, \tag{66}$$

$$\eta_j^{\pm} = a_j\left(\beta_2 - \frac{\alpha_{12}\beta_1}{\alpha_{11}}\right) \pm \eta, \tag{67}$$

$$\tau_j^{\pm} = a_j\left(1 + \frac{\beta_1^2}{\alpha_{11}}\right) \pm \tau, \tag{68}$$

$$\eta_0^{*} = a_0\left(\beta_2 - \frac{\alpha_{12}\beta_1}{\alpha_{11}}\right), \tag{69}$$

$$\tau_0^{*} = a_0\left(1 + \frac{\beta_1^2}{\alpha_{11}}\right). \tag{70}$$

The operator $\mathcal{B}_0^+$ (obtained by setting $j = 0$ in (62)) is identical to the operator $\mathcal{B}_0$ given by (38). It can be shown that (58)–(61) are equivalent to the boundary condition

$$\mathcal{B}_0^+\left[\prod_{j=1}^{P}(\mathcal{B}_j^+)^2\right]\left[\prod_{j=1}^{Q}(\Sigma_j)^2\right]u = 0 \quad \text{on } \Gamma_E, \tag{71}$$

which extends (57) to the general case.

Now we make a few remarks concerning these recursive relations.

Remark 1. As mentioned previously, the parameters $0 < a_j \leqslant 1$ may be chosen in several ways [19]. The simplest is the "Padé choice" $a_j = 1$ for all j with no evanescent mode terms ($Q = 0$), which was shown in [19] to be generally a reasonable choice for short time computations. In subsequent work we will test optimized combinations of $\{a_j\}$ and $\{\sigma_j\}$. Excellent results were obtained in [20] by choosing the $\{a_j\}$ to be cosines of Gauss–Radau quadrature nodes in $\left[0, \frac{\pi}{2}\right]$ and the $\{\sigma_j\}$ to be Yarvin–Rokhlin nodes [34]. See also the discussion in [23] on the choice of values of σ_j.

Remark 2. The derivation above relates to the east edge of the boundary Γ_E. A similar derivation applies to the three other edges. Each edge is associated with its own set of auxiliary variables and recursive relations. Corner conditions are needed to connect among these different sets [6,22,20]. Corner conditions for the general wave equation should be developed by extending the corner conditions given in these references, but we shall not do so in the present paper.

Remark 3. The fact that the present formulation possesses enhanced stability compared to the Givoli–Neta condition [22] (and not only enhanced accuracy owing to the squaring in (71)) has to do with the fact that the derivatives in (49) and (50) are of uniform order in contrast to (45) and (46) [35].

Remark 4. Similarly to what has been commented on (45)–(47), the recursive relations (58)–(61) do not constitute a practical ABC themselves, because (59) involves the *normal derivative* $\partial_x\phi_j$. In the next section we shall derive a high-order ABC that is free from such derivatives, focusing for simplicity on the case $Q = 0$.

Remark 5. A reinterpretation of the auxiliary variable indices as grid indices again leads to a "PML-like" formulation. However, this formulation cannot be simply expressed in terms of frequency-dependent mesh spacings. This is not surprising as known stable formulations of PML for anisotropic systems are also more complex than in the isotropic case; see [32].

3.4. High-order ABC

We now consider the recursive relations given by (59) for $j \geqslant 1$, assuming for now that $Q = 0$. Our goal is to obtain new relations which, in contrast to (59), do not involve the normal derivative $\partial_x\phi_j$. This would lead to a practical high-order ABC. Thus, we would like to replace (58)–(61) by a condition of the form

$$\mathcal{B}_0^+ u - \mathcal{B}_0^{*}\phi_1 = 0 \quad \text{on } \Gamma_E, \tag{72}$$

$$\mathcal{H}_{j,j-1}\phi_{j-1} + \mathcal{H}_{j,j}\phi_j + \mathcal{H}_{j,j+1}\phi_{j+1} = 0, \quad j = 1,\ldots,P \quad \text{on } \Gamma_E, \tag{73}$$

$$\phi_0 \equiv u, \quad \phi_{P+1} = 0 \quad \text{on } \Gamma_E, \tag{74}$$

where $\mathcal{H}_{j,j-1}, \mathcal{H}_{j,j}$ and $\mathcal{H}_{j,j+1}$ are linear operators involving 2nd-order time and tangential derivatives, *but not the normal derivative*.

In achieving this goal it is necessary to make use of the fact that all the auxiliary variables ϕ_j satisfy the same wave Eq. (1) as u does, i.e.,

$$\mathcal{L}\phi_j = 0, \quad j = 1,\ldots,P+1. \tag{75}$$

This is proved in Appendix A. The next step is to manipulate (59) by using (75), thus getting rid of all the normal derivatives. For this purpose it is convenient to write (75) (see (1)) in the form

$$(\alpha_{11}\partial_x^2 + 2\alpha_{12}\partial_{xy} - 2\beta_1\partial_{tx})\phi_j = (-\alpha_{22}\partial_y^2 + \partial_t^2 + 2\beta_2\partial_{ty} + \nu)\phi_j. \tag{76}$$

Here we wrote on the left side all the terms which involve the normal derivative ∂_x, and on the right side all the terms that do not involve ∂_x.

Now we apply the operator $\mathcal{B}_j^+$ to both sides of the jth relation in (59), and separately we also apply $\mathcal{B}_{j-1}^-$ to the $(j-1)$th relation. The results are

$$\mathcal{B}_j^+\mathcal{B}_j^+\phi_j = \mathcal{B}_j^+\mathcal{B}_j^-\phi_{j+1}, \tag{77}$$

$$\mathcal{B}_{j-1}^-\mathcal{B}_{j-1}^+\phi_{j-1} = \mathcal{B}_{j-1}^-\mathcal{B}_{j-1}^-\phi_j. \tag{78}$$

This holds for $j \geqslant 2$. We multiply (77) by C_j and (78) by D_j, where C_j and D_j are yet undetermined constants, and we add the two equations to obtain

$$(C_j\mathcal{B}_j^+\mathcal{B}_j^+ + D_j\mathcal{B}_{j-1}^-\mathcal{B}_{j-1}^-)\phi_j = C_j\mathcal{B}_j^+\mathcal{B}_j^-\phi_{j+1} + D_j\mathcal{B}_{j-1}^-\mathcal{B}_{j-1}^+\phi_{j-1}. \tag{79}$$

Let us now consider the operator on the left side of (79). By using (62) and (63) we get

$$\begin{aligned} C_j\mathcal{B}_j^+\mathcal{B}_j^+ + D_j\mathcal{B}_{j-1}^-\mathcal{B}_{j-1}^- = {} & C_j(\eta_j^+\partial_y + \tau_j^+\partial_t)^2 + C_j\xi^2\partial_x^2 + 2C_j(\eta_j^+\partial_y + \tau_j^+\partial_t)\xi\partial_x + D_j(\eta_{j-1}^-\partial_y + \tau_{j-1}^-\partial_t)^2 + D_j\xi^2\partial_x^2 \\ & - 2D_j(\eta_{j-1}^-\partial_y + \tau_{j-1}^-\partial_t)\xi\partial_x. \end{aligned} \tag{80}$$

We would like this combination to be free from the normal derivative ∂_x. We note that the first and fourth terms on the right side of (80) are "harmless" because they do not involve ∂_x. Thus, we have to replace the second, third, fifth and sixth terms by an expression which does not involve ∂_x either. This would be possible if these terms reduce to the operator on the left side of (76), because then they can be replaced by the operator on the right side of (76) which is free of ∂_x. Thus, we require

$$C_j\xi^2\partial_x^2 + 2C_j(\eta_j^+\partial_y + \tau_j^+\partial_t)\xi\partial_x + D_j\xi^2\partial_x^2 - 2D_j(\eta_{j-1}^-\partial_y + \tau_{j-1}^-\partial_t)\xi\partial_x = \alpha_{11}\partial_x^2 + 2\alpha_{12}\partial_{xy} - 2\beta_1\partial_{tx}. \tag{81}$$

We equate coefficients of equal derivatives (∂_x^2, ∂_{xy} and ∂_{tx}) and obtain three equations for the constants C_j and D_j. These equations are linearly dependent, and it is possible to show that if C_j and D_j satisfy the ∂_x^2 equation and the ∂_{tx} equation, then they also satisfy the ∂_{xy} equation. Solving the former two equations we obtain

$$\text{For } j \geqslant 2: \quad C_j = \frac{\alpha_{11}\tau_{j-1}^- - \beta_1\xi}{(\tau_{j-1}^- + \tau_j^+)\xi^2}, \quad D_j = \frac{\alpha_{11}\tau_j^+ + \beta_1\xi}{(\tau_{j-1}^- + \tau_j^+)\xi^2}. \tag{82}$$

From (80), (81) and (76) we get

$$(C_j\mathcal{B}_j^+\mathcal{B}_j^+ + D_j\mathcal{B}_{j-1}^-\mathcal{B}_{j-1}^-)\phi_j = (C_j(\eta_j^+\partial_y + \tau_j^+\partial_t)^2 + D_j(\eta_{j-1}^-\partial_y + \tau_{j-1}^-\partial_t)^2 - \alpha_{22}\partial_y^2 + \partial_t^2 + 2\beta_2\partial_{ty} + \nu)\phi_j. \tag{83}$$

Thus we have managed to turn the left side of (79) into an expression which does not involve the normal derivative.

Now we consider the right side of (79). Simple calculation, using (62) and (63), yields

$$\mathcal{B}_j^+\mathcal{B}_j^- = (\tau_j^+\partial_t + \eta_j^+\partial_y)(\tau_j^-\partial_t + \eta_j^-\partial_y) - \xi^2\partial_x^2 + \xi[(\tau_j^- + \tau_j^+)\partial_{tx} + (\eta_j^- + \eta_j^+)\partial_{xy}]. \tag{84}$$

As before, we would like the terms in (84) that involve ∂_x to reduce to the operator on the left side of (76). Thus, we ask about the existence of a constant E_j such that

$$E_j\{-\xi^2\partial_x^2 + \xi[(\tau_j^- + \tau_j^+)\partial_{tx} + (\eta_j^- + \eta_j^+)\partial_{xy}]\} = \alpha_{11}\partial_x^2 + 2\alpha_{12}\partial_{xy} - 2\beta_1\partial_{tx}. \tag{85}$$

Of course, each of the coefficients of $\partial_x^2, \partial_{tx}$ and ∂_{xy} should be equal on both sides, for (85) to hold identically. It is easy to show that the constant E_j exists and that its value is

$$E_j = -(C_j + D_j) = -\frac{\alpha_{11}}{\xi^2}.$$

From (84), (85) and (76) we then have

$$\mathcal{B}_j^+\mathcal{B}_j^-\phi_{j+1} = \left[\left(\tau_j^+\partial_t + \eta_j^+\partial_y\right)\left(\tau_j^-\partial_t + \eta_j^-\partial_y\right) - \frac{\xi^2}{\alpha_{11}}\left(-\alpha_{22}\partial_y^2 + \partial_t^2 + 2\beta_2\partial_{ty} + \nu\right)\right]\phi_{j+1}. \tag{86}$$

Thus, we managed to write the first term on the right side of (79) such that it does not involve the normal derivative. Now, from the last result we also get immediately

$$\mathcal{B}_{j-1}^-\mathcal{B}_{j-1}^+\phi_{j-1} = \mathcal{B}_{j-1}^+\mathcal{B}_{j-1}^-\phi_{j-1} = \left[\left(\tau_{j-1}^+\partial_t + \eta_{j-1}^+\partial_y\right)\left(\tau_{j-1}^-\partial_t + \eta_{j-1}^-\partial_y\right) - \frac{\xi^2}{\alpha_{11}}\left(-\alpha_{22}\partial_y^2 + \partial_t^2 + 2\beta_2\partial_{ty} + \nu\right)\right]\phi_{j-1}. \tag{87}$$

Thus, we also managed to write the second term on the right side of (79) such that it does not involve the normal derivative.

By using (83), (86) and (87), we are now in a position to rewrite (79) such that it does not involve the normal derivative. The end result, which is the desired jth ABC equation for $j \geqslant 2$, has the form (73) with

For $j \geqslant 2$:

$$\mathcal{H}_{j,j-1} = -D_j\left[(\tau_{j-1}^+\partial_t + \eta_{j-1}^+\partial_y)(\tau_{j-1}^-\partial_t + \eta_{j-1}^-\partial_y) - \frac{\xi^2}{\alpha_{11}}(-\alpha_{22}\partial_y^2 + \partial_t^2 + 2\beta_2\partial_{ty} + v)\right], \tag{88}$$

$$\mathcal{H}_{j,j} = C_j(\eta_j^+\partial_y + \tau_j^+\partial_t)^2 + D_j(\eta_{j-1}^-\partial_y + \tau_{j-1}^-\partial_t)^2 - \alpha_{22}\partial_y^2 + \partial_t^2 + 2\beta_2\partial_{ty} + v, \tag{89}$$

$$\mathcal{H}_{j,j+1} = -C_j\left[(\tau_j^+\partial_t + \eta_j^+\partial_y)(\tau_j^-\partial_t + \eta_j^-\partial_y) - \frac{\xi^2}{\alpha_{11}}(-\alpha_{22}\partial_y^2 + \partial_t^2 + 2\beta_2\partial_{ty} + v)\right]. \tag{90}$$

It remains to find the ABC equation for $j = 1$. From (58) and (59) we have

$$\mathcal{B}_0^+ u = \mathcal{B}_0^* \phi_1, \quad \mathcal{B}_1^+\phi_1 = \mathcal{B}_1^-\phi_2. \tag{91}$$

We apply $C_1\mathcal{B}_1^+$ to the second equation and $D_1\mathcal{B}_0^-$ to the first equation and subtract them, to obtain

$$(C_1\mathcal{B}_1^+\mathcal{B}_1^+ + D_1\mathcal{B}_0^-\mathcal{B}_0^*)\phi_1 = C_1\mathcal{B}_1^+\mathcal{B}_1^-\phi_2 + D_1\mathcal{B}_0^-\mathcal{B}_0^+ u.$$

This is similar to (79) (but note that it cannot be obtained from (79) by setting $j = 1$ due to the operator $\mathcal{B}_0^*$ appearing here which is different than $\mathcal{B}_0^-$). We proceed along the same lines as in the case $j \geqslant 2$. The end result is

$$\mathcal{H}_{1,0} = -D_1\left[(\tau_0^+\partial_t + \eta_0^+\partial_y)(\tau_0^-\partial_t + \eta_0^-\partial_y) - \frac{\xi^2}{\alpha_{11}}(-\alpha_{22}\partial_y^2 + \partial_t^2 + 2\beta_2\partial_{ty} + v)\right], \tag{92}$$

$$\mathcal{H}_{1,1} = C_1(\eta_1^+\partial_y + \tau_1^+\partial_t)^2 + D_1(\eta_0^-\partial_y + \tau_0^-\partial_t)(\eta_0^*\partial_y + \tau_0^*\partial_t) - \alpha_{22}\partial_y^2 + \partial_t^2 + 2\beta_2\partial_{ty} + v, \tag{93}$$

$$\mathcal{H}_{1,2} = -C_1\left[(\tau_1^+\partial_t + \eta_1^+\partial_y)(\tau_1^-\partial_t + \eta_1^-\partial_y) - \frac{\xi^2}{\alpha_{11}}(-\alpha_{22}\partial_y^2 + \partial_t^2 + 2\beta_2\partial_{ty} + v)\right], \tag{94}$$

where

$$C_1 = \frac{\alpha_{11}}{\xi^2}, \quad D_1 = \frac{2}{\xi^2\tau_0^*}(\alpha_{11}\tau_1^+ + \xi\beta_1). \tag{95}$$

We note that (92)–(94) are analogous to (88)–(90) except for the second term on the right of (93) which involves η_0^* and τ_0^*.

The derivation of the boundary system in the presence of the evanescent mode recursions ($Q > 0$) would proceed along similar lines. Clearly the equations derived above would still hold for $j \leqslant P$. Also, for $j > P + 1$ an analogous system can be derived. A complication arises in coupling the separate propagating mode and evanescent mode subsystems. In [23] we accomplished this by adding a bridge variable. The details of including evanescent mode corrections for the general wave equation considered here will be left to subsequent work; the numerical experiments here will be carried out for $Q = 0$.

3.5. Special cases

3.5.1. Isotropic dispersive medium

In the case of isotropic stationary dispersive medium, the parameters of the wave operator $\mathcal{L}$ in (1) are restricted by $\alpha_{ij} = c^2\delta_{ij}$ and $\beta_1 = \beta_2 = 0$. In this case $\xi = c, \eta_0^* = \eta_j^- = \eta_j^+ = 0, \tau_j^+ = \tau_j^- = a_j, \tau_0^* = a_0$, and the general results above reduce to the following:

$$\mathcal{B}_j^+ = c\partial_x + a_j\partial_t, \tag{96}$$

$$\mathcal{B}_j^- = -c\partial_x + a_j\partial_t,$$

$$\mathcal{B}_0^* = a_0\partial_t,$$

$$\mathcal{H}_{1,0} = \frac{2a_1}{a_0}[(1 - a_0^2)\partial_t^2 - c^2\partial_y^2 + v], \tag{97}$$

$$\mathcal{H}_{1,1} = (a_1^2 + 2a_0a_1 + 1)\partial_t^2 - c^2\partial_y^2 + v,$$

$$\mathcal{H}_{1,2} = (1 - a_1^2)\partial_t^2 - c^2\partial_y^2 + v, \tag{98}$$

For $j \geqslant 2$:

$$\mathcal{H}_{j,j-1} = D_j[(1 - a_{j-1}^2)\partial_t^2 - c^2\partial_y^2 + v], \tag{99}$$

$$\mathcal{H}_{j,j} = (C_ja_j^2 + D_ja_{j-1}^2 + 1)\partial_t^2 - c^2\partial_y^2 + v,$$

$$\mathcal{H}_{j,j+1} = C_j[(1 - a_j^2)\partial_t^2 - c^2\partial_y^2 + v], \tag{100}$$

$$C_j = \frac{a_{j-1}}{a_{j-1} + a_j}, \quad D_j = \frac{a_j}{a_{j-1} + a_j}. \tag{101}$$

If we use these operators in (72) and (73) while multiplying the $j \geqslant 2$ equations (corresponding to (99) and (100)) by $(a_{j-1} + a_j)$ and the $j = 1$ equation (corresponding to (97) and (98)) by a_0, we reproduce the ABC obtained in [23]:

$$(a_0\partial_t + c\partial_x)u = a_0\partial_t\phi_1, \tag{102}$$

$$l_{j,j-1}\partial_t^2\phi_{j-1} + l_{j,j}\partial_t^2\phi_j + l_{j,j+1}\partial_t^2\phi_{j+1} = c^2(m_{j,j-1}\partial_y^2\phi_{j-1} + m_{j,j}\partial_y^2\phi_j + m_{j,j+1}\partial_y^2\phi_{j+1}) - v(m_{j,j-1}\phi_{j-1} + m_{j,j}\phi_j + m_{j,j+1}\phi_{j+1}), \quad j = 1,\dots,P, \tag{103}$$

$$\phi_0 \equiv u, \quad \phi_{P+1} = 0, \tag{104}$$

where the coefficients for $j = 1$ are

$$l_{1,0} = 2a_1(1 - a_0^2), \quad l_{1,1} = a_0(1 + 2a_0a_1 + a_1^2), \quad l_{1,2} = a_0(1 - a_1^2), \tag{105}$$

$$m_{1,0} = 2a_1, \quad m_{1,1} = a_0, \quad m_{1,2} = a_0, \tag{106}$$

and the coefficients for $j = 2,\dots,P$ are

$$l_{j,j-1} = a_j(1 - a_{j-1}^2), \quad l_{j,j} = a_j(1 + a_{j-1}^2) + a_{j-1}(1 + a_j^2), \quad l_{j,j+1} = a_{j-1}(1 - a_j^2), \tag{107}$$

$$m_{j,j-1} = a_j, \quad m_{j,j} = a_{j-1} + a_j, \quad m_{j,j+1} = a_{j-1}. \tag{108}$$

If in addition we set $v = 0$ we reproduce the original basic Hagstrom–Warburton ABC for the classical wave equation [6].

3.5.2. Anisotropic medium

In the case of anisotropic stationary medium, the parameters of the wave operator $\mathcal{L}$ in (1) are restricted by $\beta_1 = \beta_2 = 0$. In this case $\xi = \sqrt{\alpha_{11}}, \tau_j^+ = \tau_j^- = a_j, \eta_j^+ = \alpha_{12}/\sqrt{\alpha_{11}}, \eta_j^- = -\alpha_{12}/\sqrt{\alpha_{11}}, \tau_0^* = a_0, \eta_0^* = 0$, and C_j and D_j are the same as in (101). Then it can be shown that

$$\mathcal{B}_j^+ = \sqrt{\alpha_{11}}\partial_x + \left(\alpha_{12}/\sqrt{\alpha_{11}}\right)\partial_y + a_j\partial_t, \tag{109}$$

$$\mathcal{B}_j^- = -\sqrt{\alpha_{11}}\partial_x - \left(\alpha_{12}/\sqrt{\alpha_{11}}\right)\partial_y + a_j\partial_t, \tag{110}$$

$$\mathcal{B}_0^* = a_0\partial_t. \tag{111}$$

This implies, after some algebra, that the first ABC equation is

$$\left(a_0\partial_t + \sqrt{\alpha_{11}}\partial_x + \frac{\alpha_{12}}{\sqrt{\alpha_{11}}}\partial_y\right)u = a_0\partial_t\phi_1, \tag{112}$$

and that the formulas (97)–(101) as well as (103)–(108) hold in the anisotropic case too (with $v = 0$ if there is no dispersion), but instead of the isotropic definition of the wave speed $c^2 = \alpha_{11} = \alpha_{22}$ we have here the "effective wave speed" $c = c_e$, where

$$c_e^2 \equiv \alpha_{22}\left(1 - \frac{\alpha_{12}^2}{\alpha_{11}\alpha_{22}}\right) = \frac{\alpha_{11}\alpha_{22} - \alpha_{12}^2}{\alpha_{11}}. \tag{113}$$

It is interesting to note that in the orthotropic case ($\alpha_{12} = 0$) two different wave speeds are involved in this ABC: $\sqrt{\alpha_{11}}$ which appears in (112) and the effective speed c_e defined above which appears in (103).

3.5.3. Convective medium

In the case of a non-dispersive isotropic medium moving with constant subsonic velocity $\boldsymbol{V} = \{V_i\}$, the parameters of the wave operator $\mathcal{L}$ in (1) are restricted by $\alpha_{11} = c^2 - V_1^2,\ \alpha_{12} = -V_1V_2,\ \alpha_{22} = c^2 - V_2^2,\ \beta_1 = V_1, \beta_2 = V_2,\ v = 0$, where c is the wave speed. For convenience we define the Mach numbers

$$M_1 = \frac{V_1}{c}, \quad M_2 = \frac{V_2}{c}. \tag{114}$$

Since the flow is subsonic, $|M_1| < 1$ and $|M_2| < 1$. Then $\xi = c, \tau_j^+ = (a_j - M_1)/(1 - M_1^2),\ \tau_j^- = (a_j + M_1)/(1 - M_1^2),\ \eta_j^+ = V_2\tau_j^+,\ \eta_j^- = V_2\tau_j^-,\ \tau_0^* = a_0/(1 - M_1^2),\ \eta_0^* = V_2\tau_0^*,\ C_j = (1 - M_1^2)a_{j-1}/(a_j + a_{j-1})$ and $D_j = (1 - M_1^2)a_j/(a_j + a_{j-1})$. This leads to

$$\mathcal{B}_j^+ = c\partial_x + [(a_j - M_1)/(1 - M_1^2)](V_2\partial_y + \partial_t),$$

$$\mathcal{B}_j^- = -c\partial_x + [(a_j + M_1)/(1 - M_1^2)](V_2\partial_y + \partial_t),$$

$$\mathcal{B}_0^* = [a_0/(1 - M_1^2)](V_2\partial_y + \partial_t),$$

$$\mathcal{H}_{1,0} = \frac{2a_1}{a_0}\left[\frac{M_1^2 - a_0^2}{1 - M_1^2}(\partial_t + V_2\partial_y)^2 - (c^2 - V_2^2)\partial_y^2 + 2V_2\partial_{ty} + \partial_t^2\right],$$

$$\mathcal{H}_{1,1} = \frac{a_1^2 + 2a_0a_1 + M_1^2}{1 - M_1^2}(\partial_t + V_2\partial_y)^2 + \partial_t^2 + 2V_2\partial_{ty} - (c^2 - V_2^2)\partial_y^2,$$

$$\mathcal{H}_{1,2} = \frac{M_1^2 - a_1^2}{1 - M_1^2}(\partial_t + V_2\partial_y)^2 - (c^2 - V_2^2)\partial_y^2 + 2V_2\partial_{ty} + \partial_t^2,$$

For $j \geqslant 2$:

$$\mathcal{H}_{j,j-1} = \widetilde{D}_j \left[\frac{M_1^2 - a_{j-1}^2}{1 - M_1^2} (\partial_t + V_2 \partial_y)^2 - (c^2 - V_2^2)\partial_y^2 + 2V_2 \partial_{ty} + \partial_t^2 \right],$$

$$\mathcal{H}_{j,j} = \frac{\widetilde{C}_j (a_j - M_1)^2 + \widetilde{D}_j (a_{j-1} + M_1)^2}{1 - M_1^2} (\partial_t + V_2 \partial_y)^2 - (c^2 - V_2^2)\partial_y^2 + 2V_2 \partial_{ty} + \partial_t^2,$$

$$\mathcal{H}_{j,j+1} = \widetilde{C}_j \left[\frac{M_1^2 - a_j^2}{1 - M_1^2} (\partial_t + V_2 \partial_y)^2 - (c^2 - V_2^2)\partial_y^2 + 2V_2 \partial_{ty} + \partial_t^2 \right],$$

$$\widetilde{C}_j = \frac{a_{j-1}}{a_{j-1} + a_j}, \quad \widetilde{D}_j = \frac{a_j}{a_{j-1} + a_j}.$$

4. Reflection coefficient

4.1. Basic reflection coefficient

We consider the basic ABCs (38), (43) and derive the reflection coefficient that they generate when a plane-wave impinges on the boundary Γ_E. Since the operator used in the high-order ABC is a product of operators of the basic form (cf. (71)), it is required that the reflection coefficient R_0 of the basic ABC satisfy $|R_0| < 1$ for all possible plane waves. In fact, in contrast to the case with traditional low-order ABCs, in the present case it is not required that $|R_0|$ be very small, since the high-order ABC has the effect of raising $|R_0|$ to a high power.

We take a propagating plane-wave of the form (cf. (12) and (26))

$$u = u_I + u_R = \exp[i\omega(t - K_x^I x - K_y y)] + R_0 \exp[i\omega(t - K_x^R x - K_y y)]. \tag{115}$$

Here the two terms represent the incident and reflected waves, respectively, and R_0 is the reflection coefficient. The same K_y is taken for the incident and reflected waves; this must be the case so that the exponents in y can be balanced when a homogeneous boundary condition is imposed on Γ_E. Note that

$$K_x^I = K_x^+, \quad K_x^R = K_x^-,$$

where K_x^+ and K_x^- are the two roots of K_x obtained from the dispersion relation; see Figs. 2(b) and 3(b). To find the relation between K_x^R and K_x^I we use (14) to find

$$K_x^I = \frac{1}{\alpha_{11}} \left[-(\alpha_{12} K_y + \beta_1) + \sqrt{\Delta} \right], \quad K_x^R = \frac{1}{\alpha_{11}} \left[-(\alpha_{12} K_y + \beta_1) - \sqrt{\Delta} \right]. \tag{116}$$

Adding these equations together yields the relation

$$K_x^R = -K_x^I - \frac{2(\alpha_{12} K_y + \beta_1)}{\alpha_{11}}. \tag{117}$$

Now we substitute the plane-wave (115) in the basic ABC (38) to get

$$|R_0| = \frac{|-\xi K_x^I - \eta_0 K_y + \tau_0|}{|-\xi K_x^R - \eta_0 K_y + \tau_0|}. \tag{118}$$

Substituting (117) in this expression as well as (40) and (41) and noting (37) yields

$$|R_0| = \frac{|a_0 - a|}{|a_0 + a|}. \tag{119}$$

Since $0 < a_0, a \leqslant 1$ it is obvious that $|R_0| < 1$. This is equivalent to Higdon's expression for $|R|$ in the isotropic case [11] with $a = \cos\theta$ (θ being the angle of incidence). See the discussion around Eq. (36).

If, on the other hand, we consider an incoming and reflected evanescent wave we have:

$$u = u_I + u_R = \exp[i\omega t - \gamma^+ x - ik_2 y] + S_0 \exp[i\omega t - \gamma^- x - ik_2 y], \tag{120}$$

with (see (25))

$$\gamma^\pm = -\frac{i}{\alpha_{11}} (\alpha_{12} k_2 + \beta_1) \pm \sigma.$$

We now obtain a complex reflection coefficient:

$$|S_0| = \frac{|-\xi\gamma^+ - i\eta_0 k_2 + i\tau_0\omega|}{|-\xi\gamma^- - i\eta_0 k_2 + i\tau_0\omega|} = 1. \tag{121}$$

Thus evanescent modes are perfectly reflected. The same calculations can be carried out for the basic evanescent mode operator (43), with exactly the opposite results. In particular, denoting the reflection coefficients by $\widetilde{R}_0$ and $\widetilde{S}_0$ we find:

$$|\widetilde{R}_0| = \frac{|\sigma_0 - iaD_1 \frac{\sqrt{\alpha_{11}+\beta_1^2}}{\alpha_{11}}|}{|\sigma_0 + iaD_1 \frac{\sqrt{\alpha_{11}+\beta_1^2}}{\alpha_{11}}|} = 1, \quad |\widetilde{S}_0| = \frac{|\sigma_0 - \sigma|}{|\sigma_0 + \sigma|} < 1.$$

Thus, each type of operator controls the reflection coefficient for the corresponding type of mode. With an ABC which includes both types of operators, both propagating and decaying modes are controlled.

4.2. High-order reflection coefficient

Using (71), it is easy to generalize the result (119) to the high-order ABC. Analogously to (118) we assume an outgoing propagating plane-wave and define

$$|R_j| = \frac{|-\xi K_x^I - \eta_j^+ K_y + \tau_j^+|}{|-\xi K_x^R - \eta_j^+ K_y + \tau_j^+|}. \tag{122}$$

Owing to (71), it is easy to see that the total reflection coefficient for the Pth-order ABC is

$$|R| = |R_0| \prod_{j=1}^{P} |R_j|^2. \tag{123}$$

As we have shown above, each factor in (123) is smaller than 1; hence $|R|$ approaches zero exponentially fast as P is increased. For example, if $a_j = 1$ for all j, (123) reduces to

$$|R| = |R_0|^{2P+1}. \tag{124}$$

To gain more insight, we derive this result more directly for the anisotropic wave Eq. (7) with $a_j = 1$. We consider the recursive relations (58)–(61), with the operators (109)–(111). We assume that all the auxiliary variables ϕ_j have the plane-wave form (cf. (115))

$$\phi_j = \mu_j \exp[i(\omega t - k_x^I x - k_y y)] + \rho_j \exp[i(\omega t - k_x^R x - k_y y)], \quad j = 0, \ldots, P+1. \tag{125}$$

Here $\phi_0 \equiv u, \mu_0 = 1$ and $\rho_0 = R$, which is the reflection coefficient. We denote

$$E^I = \sqrt{\alpha_{11}} k_x^I + \frac{\alpha_{12}}{\sqrt{\alpha_{11}}} k_y, \quad E^R = \sqrt{\alpha_{11}} k_x^R + \frac{\alpha_{12}}{\sqrt{\alpha_{11}}} k_y.$$

Now we substitute (125) into (58)–(61), and without loss of generality we set $x = 0$ as the location of Γ_E. This yields

$$\mu_0(\omega - E^I) + \rho_0(\omega - E^R) = \mu_1 \omega + \rho_1 \omega, \tag{126}$$

$$\mu_j(\omega - E^I) + \rho_j(\omega - E^R) = \mu_{j+1}(\omega + E^I) + \rho_{j+1}(\omega + E^R), \quad j = 1, \ldots, P, \tag{127}$$

$$\mu_{P+1} + \rho_{P+1} = 0. \tag{128}$$

It seems that there are too many unknowns in Eqs. (126)–(128). However, we can make the following argument. Eqs. (58) and (59) can be differentiated r times with respect to x, since they hold in the entire domain D_E and not only on Γ_E. This would introduce the coefficients $(-k_x^I)^r$ and $(-k_x^R)^r$ into Eqs. (126) and (127). Since these equations must hold *for any* r, we must conclude that the μ-terms and the ρ-terms in (126) and (127) are self-balanced *separately*. Thus, we may replace (126)–(128) by

$$\begin{aligned}
&\mu_0(\omega - E^I) = \mu_1 \omega, \\
&\rho_0(\omega - E^R) = \rho_1 \omega, \\
&\mu_j(\omega - E^I) = \mu_{j+1}(\omega + E^I), \quad j = 1, \ldots, P, \\
&\rho_j(\omega - E^R) = \rho_{j+1}(\omega + E^R), \quad j = 1, \ldots, P, \\
&\mu_{P+1} + \rho_{P+1} = 0.
\end{aligned}$$

Now we use these equations recursively to calculate R:

$$\begin{aligned}
R = \rho_0 &= \rho_1 \frac{\omega}{\omega - E^R} = \rho_2 \frac{\omega + E^R}{\omega - E^R} \frac{\omega}{\omega - E^R} = \ldots = \rho_{P+1} \left(\frac{\omega + E^R}{\omega - E^R}\right)^P \frac{\omega}{\omega - E^R} = -\mu_{P+1} \left(\frac{\omega + E^R}{\omega - E^R}\right)^P \frac{\omega}{\omega - E^R} \\
&= -\mu_P \frac{\omega - E^I}{\omega + E^I} \left(\frac{\omega + E^R}{\omega - E^R}\right)^P \frac{\omega}{\omega - E^R} = \ldots = -\mu_1 \left(\frac{\omega - E^I}{\omega + E^I}\right)^P \left(\frac{\omega + E^R}{\omega - E^R}\right)^P \frac{\omega}{\omega - E^R} = -\frac{\omega - E^I}{\omega - E^R} \left(\frac{\omega - E^I}{\omega + E^I}\right)^P \left(\frac{\omega + E^R}{\omega - E^R}\right)^P.
\end{aligned}$$

However, it is easy to show (see (116) and (117)) that $E^I = -E^R > 0$. Hence we have

$$R = -\left(\frac{\omega - E^I}{\omega + E^I}\right)^{2P+1}, \tag{129}$$

which is the desired result. It is easy to check that (122) reduces to the quantity in parentheses in (129) in the anisotropic case with $a_j = 1$.

Similar analyses hold for products of the basic evanescent mode operators, S_j, and for outgoing evanescent modes. In particular, the reflection coefficient for a product of propagating mode operators applied to an evanescent mode has modulus one, as does a product of evanescent mode operators applied to a propagating mode. Products of evanescent mode operators applied to evanescent modes are also exponentially convergent with increasing order.

Note that boundary conditions based on propagating modes alone are, in fact, convergent for finite times in the isotropic case, though this requires more subtle analysis. See [1,21]. Nonetheless, the convergence is poor for long times. For (almost) uniform convergence in time the evanescent mode corrections are needed [20].

5. Well-posedness

5.1. Basic ABC: well-posedness via the Kreiss-criterion

The Kreiss-criterion [36] for well-posedness (which is related to stability on the continuous level) has several equivalent versions; see, e.g., [35] and [37–40]. Roughly speaking, the Kreiss-criterion states that a problem consisting of a linear strictly-hyperbolic equation (or a system of equations) in Ω, boundary conditions on Γ and initial conditions is well-posed if and only if it does not admit as solutions any plane waves which spatially decay into Ω and grow in time. More precisely, considering the problem in the left half-plane with boundary Γ_E as in Fig. 4, the Kreiss-criterion looks at inadmissible solutions of the form (12) which belong to two categories, i.e.,

$$\text{Type-1 solutions}: \quad \text{Im } k_2 = 0, \quad \text{Im } k_1 < 0, \quad \text{Im } \omega > 0, \tag{130}$$

$$\text{Type-2 solutions}: \quad \text{Im } k_2 = 0, \quad \text{Im } \omega > 0, \quad \text{Im } \omega \to 0, \tag{131}$$

$$\text{a}: \quad \text{Im } k_1 < 0, \tag{132}$$

$$\text{b}: \quad \text{Im } k_1 < 0, \quad \text{Im } k_1 \to 0. \tag{133}$$

For a differential equation which is strictly hyperbolic [35], like (1), the Kreiss-criterion gives us the following classification. If there are no inadmissible solutions of either type-1 or type-2 then the problem is said to be *strongly well-posed*. In this case, discretizations can be devised which lead to a well-behaved numerical solution in the sense of Gustafsson, Kreiss and Sundström (see [35,39]). If there are no solutions of type-1 but there are possibly solutions of type-2, the problem is said to be *weakly well-posed*. In this case the numerical solution may exhibit polynomial growth in time. If there exist solutions of type-1, then the problem is *ill-posed*.

We consider the general wave Eq. (1) in the left half-plane, with the basic ABC (38) on Γ_E and with given initial conditions. We substitute the plane-wave form (12) in (1) and obtain the dispersion equation given by (13). We substitute (12) in the ABC (38) and obtain

$$\xi k_1 + \eta_0 k_2 - \tau_0 \omega = 0. \tag{134}$$

Now we have to see if (13) and (134) together allow any type-1 or type-2 solutions, by checking the conditions (130)–(133). It is convenient to write

$$k_1 = k_{1R} + i k_{1I}, \quad \omega = \omega_R + i\omega_I, \tag{135}$$

where the subscripts R and I stand for the real and imaginary parts, respectively. We note that for the inadmissible waves k_2 is always real, by (130) and (131).

As a preparatory step, we take the real and imaginary part of the dispersion relation (13) to obtain

$$\alpha_{11}(k_{1R}^2 - k_{1I}^2) + 2(\alpha_{12}k_2 + \beta_1\omega_R)k_{1R} - 2\beta_1\omega_I k_{1I} + \alpha_{22}k_2^2 + 2\beta_2 k_2\omega_R + v - \omega_R^2 + \omega_I^2 = 0, \tag{136}$$

$$(\alpha_{11}k_{1R} + \alpha_{12}k_2 + \beta_1\omega_R)k_{1I} + (\beta_1 k_{1R} + \beta_2 k_2 - \omega_R)\omega_I = 0. \tag{137}$$

We also take the real and imaginary part of (134) and obtain, using the definitions (39)–(41), after simplification,

$$\xi(\omega_R\beta_1 + \alpha_{12}k_2 + \alpha_{11}k_{1R}) + a_0(\lambda_2 k_2 - \xi^2\omega_R) = 0, \tag{138}$$

$$\xi(\omega_I\beta_1 + \alpha_{11}k_{1I}) - a_0\xi^2\omega_I = 0. \tag{139}$$

We first consider *type-1 solutions*, which satisfy (130). We distinguish between two cases: $a_0 = 1$ and $0 < a_0 < 1$. For $a_0 = 1$, Eq. (139) gives

$$\left(\sqrt{\alpha_{11}+\beta_1^2}-\beta_1\right)\omega_I = \alpha_{11}k_{1I}. \tag{140}$$

The quantity in the parentheses is positive; hence we cannot have $\omega_I > 0$ and $k_{1I} < 0$ as (130) requires, and thus type-1 solutions do not exist. If $0 < a_0 < 1$, then we perform the following calculation. From (138) and (139), we get, respectively,

$$k_{1R} = \frac{a_0\xi^2 - \beta_1\xi}{\alpha_{11}\xi}\omega_R - \frac{a_0\lambda_2 + \alpha_{12}\xi}{\alpha_{11}\xi}k_2, \tag{141}$$

$$k_{1I} = -\frac{\beta_1 - a_0\xi}{\alpha_{11}}\omega_I. \tag{142}$$

We use these equations to eliminate k_{1R} and k_{1I} in (137), which yields, after simplification

$$\omega_I(\lambda_2 k_2 - \xi^2\omega_R)(1-a_0^2) = 0. \tag{143}$$

For solutions of type-1 we have $\omega_I > 0$ and since $a_0 < 1$ we must conclude from (143) that

$$\omega_R = \frac{\lambda_2}{\xi^2}k_2. \tag{144}$$

Using this relation in (141) yields

$$k_{1R} = -\frac{\alpha_{12}+\beta_1\beta_2}{\xi^2}k_2. \tag{145}$$

Substituting (142), (144) and (145) in (136) gives

$$\frac{Q}{\xi^2}k_2^2 + (1-a_0^2)\frac{\xi^2}{\alpha_{11}}\omega_I^2 + \nu = 0, \tag{146}$$

where

$$Q = (\alpha_{22}+\beta_2^2)(\alpha_{11}+\beta_1^2) - (\alpha_{12}+\beta_1\beta_2)^2. \tag{147}$$

It is easy to see that $Q > 0$, and thus the three terms on the left side of (146) are non-negative. Clearly, (146) cannot hold, even if $\nu = 0$, since it implies $\omega_I = 0$ which contradicts the condition $\omega_I > 0$ of a type-1 solution. Therefore, type-1 solutions are excluded for any $0 < a_0 \leqslant 1$, which guarantees that the problem is at least weakly well-posed.

Now we consider *type-2a solutions*, which satisfy (131) and (132). Considering (140) and setting $\omega_I = 0$, we get $k_{1I} = 0$, which contradicts the requirement $k_{1I} < 0$ in (132). Thus, there is no solution of type 2a.

Finally we consider *type-2b solutions*. In this case (134) becomes a real equation, and thus we can write (138) as

$$\xi(\omega\beta_1 + \alpha_{12}k_2 + \alpha_{11}k_1) + a_0(\lambda_2 k_2 - \xi^2\omega) = 0. \tag{148}$$

Since by (131) and (133) ω_I approaches zero from above and k_{1I} approaches zero from below, we deduce that the coefficients of ω_I and k_{1I} in (137) must have the same sign, namely

$$P \equiv (\alpha_{11}k_1 + \alpha_{12}k_2 + \beta_1\omega)(\beta_1 k_1 + \beta_2 k_2 - \omega) > 0. \tag{149}$$

Our goal in the following calculation is to obtain a simple expression for P and verify that $P > 0$ cannot be satisfied, thus excluding type-2b solutions.

We find it convenient to define the following constants:

$$\Omega_1 = \omega - \boldsymbol{\beta}\cdot\mathbf{k}, \quad \Omega_2 = \alpha_{11}k_1 + \alpha_{12}k_2 + \beta_1\omega, \tag{150}$$

$$\Omega_3 = \Omega_2 - \beta_1\Omega_1 \equiv \xi^2 k_1 + \lambda_3 k_2, \quad \lambda_3 = \alpha_{12} + \beta_1\beta_2. \tag{151}$$

With this notation, (148) and (149) can be written, respectively, as

$$\xi\Omega_2 + a_0(\lambda_2 k_2 - \xi^2\omega) = 0, \tag{152}$$

$$P = -\Omega_1\Omega_2 > 0. \tag{153}$$

From (151) we get

$$k_1 = \frac{\Omega_3 - \lambda_3 k_2}{\xi^2}. \tag{154}$$

From (150) and (154) we have

$$\omega = \frac{\alpha_{11}\Omega_1 + \beta_1\Omega_2 + \lambda_2 k_2}{\xi^2}. \tag{155}$$

Using (152) and (155) we get

$$\Omega_1 = \frac{1}{a_0\alpha_{11}}(\xi - a_0\beta_1)\Omega_2. \tag{156}$$

Eqs. (153) and (156) finally yield

$$P = -\Omega_1\Omega_2 = -\frac{1}{a_0\alpha_{11}}(\xi - a_0\beta_1)\Omega_2^2.$$

Thus $P > 0$ implies

$$a_0\beta_1 - \sqrt{\alpha_{11} + \beta_1^2} > 0. \tag{157}$$

Clearly, this cannot hold since we always have $a_0\beta_1 < \sqrt{\alpha_{11} + \beta_1^2}$.

The conclusion from the analysis above is summarized by the following Theorem:

Theorem 5.1. *Consider the general wave* Eq. (1) *in the left half-plane with the ABC* (38) *on* Γ_E, *and with given initial conditions with compact support in the left half-plane. This problem is strongly well-posed.*

To analyze the basic evanescent mode conditions we simply note that the real term σ_0 is a lower order term which may be neglected in the analysis of well-posedness. Thus this case may be treated as above by setting $a_0 = 0$. The analysis of type-1 and type-2a solutions is unchanged. Considering type-2b solutions we simply note that now $\Omega_2 = 0$ and thus $P = 0$.

5.2. Basic ABC: energy estimates

Energy estimates constitute a well-known tool to prove uniqueness and stability of partial differential equations. In [41], Ha-Duong and Joly showed that *strong well-posedness* for the hyperbolic initial-boundary value problem in the sense of Kreiss can be established by constructing an "energy" function which decays in time. Such an energy estimate provides a more direct sense of stability than the Kreiss-criterion, and in fact it is stronger than the Kreiss criterion in that it leads to estimates which are uniform in time [41]. The energy is not necessarily the physical one; it merely has to be a positive quadratic form in u and its derivatives, and must be equal to zero if and only if u is identically zero (given zero initial conditions).

We consider the general wave Eq. (1) and define *two* energies, which will give us different results. The energies are:

$$E_0(t) = \frac{1}{2}\int_\Omega [(\partial_t u)^2 + \boldsymbol{\alpha}\nabla u\cdot\nabla u + vu^2]\,d\Omega, \tag{158}$$

$$E_1(t) = \frac{1}{2}\int_\Omega [(D_t u)^2 + \boldsymbol{\kappa}\nabla u\cdot\nabla u + vu^2]\,d\Omega. \tag{159}$$

Here $\boldsymbol{\alpha}$ and $\boldsymbol{\kappa}$ are the symmetric positive definite tensors discussed in Section 2 (see Eqs. (1)–(6)), and

$$D_t = \partial_t + \boldsymbol{\beta}\cdot\nabla. \tag{160}$$

Clearly, both $E_0(t)$ and $E_1(t)$ are legitimate energies. In Appendix B we prove the following:

$$\frac{dE_0}{dt}(t) = \int_\Gamma [\boldsymbol{\alpha}\nabla u\cdot\boldsymbol{n} - \boldsymbol{\beta}\cdot\boldsymbol{n}\partial_t u]\partial_t u\,d\Gamma, \tag{161}$$

$$\frac{dE_1}{dt}(t) = \int_\Gamma \left[\boldsymbol{\kappa}\nabla u\cdot\boldsymbol{n}D_t u - \frac{\boldsymbol{\beta}\cdot\boldsymbol{n}}{2}\left((D_t u)^2 + \boldsymbol{\kappa}\nabla u\cdot\nabla u + vu^2\right)\right]d\Gamma. \tag{162}$$

Here $\boldsymbol{n}$ is the unit outward vector normal to Γ. We concentrate on the east boundary Γ_E, for which $n_x = 1$, $n_y = 0$. Then (161) and (162) become

$$\frac{dE_0}{dt}(t) = \int_{\Gamma_E} [\alpha_{11}\partial_x u + \alpha_{12}\partial_y u - \beta_1\partial_t u]\partial_t u\,d\Gamma, \tag{163}$$

$$\frac{dE_1}{dt}(t) = \int_{\Gamma_E} \left[(\kappa_{11}\partial_x u + \kappa_{12}\partial_y u)D_t u - \frac{\beta_1}{2}\left((D_t u)^2 + \boldsymbol{\kappa}\nabla u\cdot\nabla u + vu^2\right)\right]d\Gamma. \tag{164}$$

Now we check under what conditions $dE_0/dt \leqslant 0$ or $dE_1/dt \leqslant 0$ hold. We start from E_0. Substituting the basic ABC (38) in (163) we obtain

$$\frac{dE_0}{dt}(t) = -\int_{\Gamma_E} \frac{a_0}{\xi}\left[\xi^2(\partial_t u)^2 + \lambda_2(\partial_y u)(\partial_t u)\right]d\Gamma.$$

If $\lambda_2 = 0$ then we get

$$\frac{dE_0}{dt}(t) = -\int_{\Gamma_E} a_0\xi(\partial_t u)^2\,d\Gamma \leqslant 0. \tag{165}$$

This establishes the strong well-posedness and stability of the problem. The condition $\lambda_2 \equiv \beta_2\alpha_{11} - \beta_1\alpha_{12} = 0$ is satisfied, for example, in the following two cases: (a) stationary anisotropic medium ($\beta_1 = \beta_2 = 0$); (b) convective orthotropic medium with background flow normal to the boundary ($\alpha_{12} = 0, \beta_2 = 0$).

Now we consider E_1. We observe that the quantity in parentheses multiplying $\beta_1/2$ in (164) is positive. We assume that $\beta_1 \geqslant 0$ from the outset, and thus the second term in the integrand of (164) is non-positive. Thus it remains to control the first term,

$$\left(\frac{dE_1}{dt}\right)_1(t) = \int_{\Gamma_E} (\kappa_{11}\partial_x u + \kappa_{12}\partial_y u) D_t u \, d\Gamma. \tag{166}$$

Using (38) and the definition (160), we get

$$D_t u = \left(\beta_1 - \frac{\xi}{\tau_0}\right)\partial_x u + \left(\beta_2 - \frac{\eta_0}{\tau_0}\right)\partial_y u.$$

Now we restrict ourselves to the case $a_0 = 1$. Simple calculation shows that

$$\beta_1 - \frac{\xi}{\tau_0} = -\sqrt{\kappa_{11}}, \quad \beta_2 - \frac{\eta_0}{\tau_0} = -\frac{\kappa_{12}}{\sqrt{\kappa_{11}}}. \tag{167}$$

Hence (166) and (167) give

$$\left(\frac{dE_1}{dt}\right)_1(t) = -\frac{1}{\sqrt{\kappa_{11}}} \int_{\Gamma_E} \left(\kappa_{11}\partial_x u + \kappa_{12}\partial_y u\right)^2 d\Gamma \leqslant 0.$$

Thus, if $a_0 = 1$ and $\beta_1 \geqslant 0$ the problem is strongly well-posed and stable.

Thus, *based on the energy estimates alone* we conclude that the problem is strongly well-posed in either of the following cases:

- $\beta_1 = \beta_2 = 0$;
- $\alpha_{12} = 0, \beta_2 = 0$;
- $a_0 = 1$ and $\beta_1 \geqslant 0$.

We believe (based on Theorem 5.1 and [41]) that these constraints are not necessary, and are the result of the limitation of our proof. Namely, we believe that it should be possible to define an appropriate energy that would decay in time for any parameters $\boldsymbol{\alpha}, \boldsymbol{\beta}, \nu$ and a_0.

5.3. Extension to high-order ABC

The extension of the Kreiss-criterion analysis of Section 5.1 to the high-order case is immediate. In Section 5.1 we substituted the plane-wave form (12) into the wave equation and into the basic ABC, obtained algebraic equations and checked whether these equations have solutions of type-1 or type-2. The algebraic equation for the basic ABC is (134). We can write it in the form

$$G(\omega, k_1, k_2; a_0) = 0, \tag{168}$$

which is exactly the form of the basic ABC (38), i.e.,

$$G(-\partial_t, \partial_x, \partial_y; a_0)u = 0 \quad \text{on } \Gamma_E. \tag{169}$$

The algebraic Eq. (168) is obtained from the Fourier transform of (169). Now, according to (71), the high-order ABC is equivalent to

$$G(-\partial_t, \partial_x, \partial_y; a_0)\left[\prod_{j=1}^{P} G^2(-\partial_t, \partial_x, \partial_y; a_j)\right] u = 0 \quad \text{on } \Gamma_E.$$

Thus, in the high-order case the algebraic Eq. (168) is replaced by the equation

$$G(\omega, k_1, k_2; a_0)\left[\prod_{j=1}^{P} G^2(\omega, k_1, k_2; a_j)\right] = 0. \tag{170}$$

At least one factor in (170) must vanish, for a certain $a_j, j \in \{0, \ldots, P\}$. We denote this a_j by a^*, and thus we have

$$G(\omega, k_1, k_2; a^*) = 0,$$

which has exactly the same form as (168). In this light, the Kreiss analysis of Section 5.1 remains unchanged, and the conclusion from this analysis thus holds in the high-order case. Therefore, we have the following.

Theorem 5.2. *Consider the general wave Eq. (1) in the left half-plane with the Pth-order ABC on Γ_E given by (58)–(61) or (71) or (72)–(74) , and with given initial conditions with compact support in the left half-plane. This problem is strongly well-posed.*

Extending the energy estimates of Section 5.2 to the high-order case is much more difficult. For the classical wave equation and a class of high-order conditions, energy estimates were derived by Ha-Duong and Joly [41]. Analogous treatment of the general wave Eq. (1) seems more complicated and will not be treated here.

6. Finite element formulation

For simplicity of the presentation we shall consider here only the Finite Element (FE) formulation for the *anisotropic wave equation* and restrict ourselves to boundary operators based on propagating modes only. The numerical experiments in the next section are based on this formulation.

We consider the wave Eq. (7) in a rectangular finite domain Ω (see Fig. 1), and the ABC on Γ_E given by (112) and (103)–(108), with the special definition (113) for $c = c_e$. To allow us to concentrate on the single artificial boundary Γ_E and avoid the issue of corner conditions for the ϕ_j, we assume (artificially) that on the west boundary Γ_W the boundary condition $u = 0$ is given, and that on the north and south boundaries homogeneous natural (in the variational sense) boundary conditions are prescribed for u and for all the ϕ_j. Initial conditions are given for u as in (10), and zero initial conditions are prescribed for the ϕ_j, as in (48).

We define the space $\mathcal{S}$ by

$$\mathcal{S} = \{w | w \in H^1(\Omega) \quad \text{and} \quad w = 0 \quad \text{on } \Gamma_W\},$$

H^1 being the Sobolev space of functions in L_2 with first derivatives in L_2. Then the weak form of the problem is:

Find $u \in \mathcal{S}$ and $\phi_j \in H^1(\Gamma_E)$ such that for all $w \in \mathcal{S}$ and all $\psi_j \in H^1(\Gamma_E)$ there holds

$$\int_\Omega w\partial_t^2 u\, d\Omega + a_0\sqrt{\alpha_{11}}\int_{\Gamma_E} w\partial_t u\, d\Gamma + \int_\Omega \nabla w\cdot\boldsymbol{\alpha}\nabla u\, d\Omega = a_0\sqrt{\alpha_{11}}\int_{\Gamma_E} w\partial_t\phi_1\, d\Gamma, \tag{171}$$

For $j = 1,\dots,P$:

$$\begin{aligned} &l_{j,j-1}\int_{\Gamma_E}\psi_j\partial_t^2\phi_{j-1}\, d\Gamma + l_{jj}\int_{\Gamma_E}\psi_j\partial_t^2\phi_j\, d\Gamma + l_{j,j+1}\int_{\Gamma_E}\psi_j\partial_t^2\phi_{j+1}\, d\Gamma \\ &+ c^2\left(m_{j,j-1}\int_{\Gamma_E}\partial_y\psi_j\partial_y\phi_{j-1}\, d\Gamma + m_{j,j}\int_{\Gamma_E}\partial_y\psi_j\partial_y\phi_j\, d\Gamma + m_{j,j+1}\int_{\Gamma_E}\partial_y\psi_j\partial_y\phi_{j+1}\, d\Gamma\right) = 0, \end{aligned} \tag{172}$$

and the initial conditions are satisfied. In these equations one should take $\phi_0 = u$ on Γ_E and $\phi_{P+1} = 0$, according to (104).

The standard spatial Galerkin FE discretization of (171) and (172) is employed. On the global-level, the variables u and ϕ_j are replaced by their finite-dimensional approximations

$$u^h(\boldsymbol{x},t) = \sum_{A=1}^{N_{h,\Omega}} d_A^h(t)N_A(\boldsymbol{x}),\ \boldsymbol{x}\in\Omega, \quad \phi_j^h(y,t) = \sum_{A=1}^{N_{h,\Gamma_E}}\phi_{jA}^h(t)N_A^{(j)}(y), \quad y\in\Gamma_E. \tag{173}$$

Here h is the mesh parameter, the index A stands for a global node number, N_A is the global-level shape function associated with the variable u^h and node A, and $N_A^{(j)}$ is the global-level shape function associated with variable ϕ_j^h and node A. Note that while N_A is a function defined in two dimensions (i.e., $N_A(x,y)$), the function $N_A^{(j)}$ is one-dimensional (i.e., $N_A^{(j)}(y)$). The expansion analogous to this on the element-level is

$$u^e(\boldsymbol{x},t) = \sum_{a=1}^{N_{en}} d_a^e(t)N_a(\boldsymbol{x}),\ \boldsymbol{x}\in\Omega^e, \quad \phi_j^e(y,t) = \sum_{a=1}^{N_{en}}\phi_{ja}^e(t)N_a^{(j)}(y), \quad y\in\Gamma_E^e. \tag{174}$$

Here Ω^e is the domain of element e, $\Gamma_E^e = \Gamma_E\cap\partial\Omega^e$, N_a is the element shape function associated with u^h and element node a, $N_a^{(j)}$ is the element shape function associated with ϕ_j^h and element node a, d_a^e is the nodal value of u^e at node a of element e, ϕ_{ja}^e is the nodal value of ϕ_j^e at node a of element e, and N_{en} is the number of element nodes. Similar expansions are used for the weight functions w and ψ_j.

In principle, different shape functions N_a and $N_a^{(j)}$ may be chosen for the different variables u and ϕ_j, for $j = 0,1,\dots,P$. We use bilinear or linear shape functions for *all* the variables, which is a most convenient choice and turns out to be stable. See [22] for discussion on the computational aspects of this choice. In the sequel we shall continue to indicate the variable number j for generality.

Using the approximations (173) in the weak Eqs. (171) and (172) leads to the following system of linear ordinary differential equations in time:

$$\boldsymbol{M}\ddot{\boldsymbol{d}} + \boldsymbol{C}\dot{\boldsymbol{d}} + \boldsymbol{K}\boldsymbol{d} = \boldsymbol{G}\dot{\boldsymbol{\phi}}_1, \tag{175}$$

$$\text{For } j = 1,\dots,P: \quad \boldsymbol{A}_j\ddot{\boldsymbol{\phi}}_{j-1} + \boldsymbol{B}_j\ddot{\boldsymbol{\phi}}_j + \boldsymbol{D}_j\ddot{\boldsymbol{\phi}}_{j+1} + \boldsymbol{E}_j\boldsymbol{\phi}_{j-1} + \boldsymbol{H}_j\boldsymbol{\phi}_j + \boldsymbol{I}_j\boldsymbol{\phi}_{j+1} = 0, \tag{176}$$

$$\boldsymbol{\phi}_0 \equiv \boldsymbol{d}|_{\Gamma_E}, \quad \boldsymbol{\phi}_{P+1} = \boldsymbol{0}, \tag{177}$$

with the initial conditions

$$\boldsymbol{d}(0) = \boldsymbol{d}_0, \quad \dot{\boldsymbol{d}}(0) = \boldsymbol{v}_0, \quad \boldsymbol{\phi}_j(0) = 0, \quad j = 1, \ldots, P. \tag{178}$$

Here a dot indicates differentiation with respect to time. The $\boldsymbol{d}$ and $\boldsymbol{\phi}_j$ are the vectors whose entries are the unknown nodal values of u in Ω and of ϕ_j on Γ_E, respectively. The first equation in (177) means that the entries of the vector $\boldsymbol{\phi}_0$ are equal to the entries of $\boldsymbol{d}$ for all the nodes on the boundary Γ_E. The element-level expressions for the arrays appearing in (175) and (176) may easily be deduced from (171) and (172):

$$M^e_{ab} = \int_{\Omega^e} N_a N_b \, d\Omega, \quad C^e_{ab} = a_0\sqrt{\alpha_{11}} \int_{\Gamma^e_E} N_a N_b \, d\Gamma, \tag{179}$$

$$K^e_{ab} = \int_{\Omega^e} \nabla N_a \cdot \boldsymbol{\alpha} \nabla N_b \, d\Omega, \quad G^e_{ab} = a_0\sqrt{\alpha_{11}} \int_{\Gamma^e_E} N_a N^{(1)}_b \, d\Gamma,$$

$$(A^e_j)_{ab} = l_{j,j-1} \int_{\Gamma^e_E} N^{(j)}_a N^{(j-1)}_b \, d\Gamma, \quad (B^e_j)_{ab} = l_{j,j} \int_{\Gamma^e_E} N^{(j)}_a N^{(j)}_b \, d\Gamma,$$

$$(D^e_j)_{ab} = l_{j,j+1} \int_{\Gamma^e_E} N^{(j)}_a N^{(j+1)}_b \, d\Gamma, \quad (E^e_j)_{ab} = c^2 m_{j,j-1} \int_{\Gamma^e_E} \partial_y N^{(j)}_a \partial_y N^{(j-1)}_b \, d\Gamma,$$

$$(H^e_j)_{ab} = c^2 m_{j,j} \int_{\Gamma^e_E} \partial_y N^{(j)}_a \partial_y N^{(j)}_b \, d\Gamma, \quad (I^e_j)_{ab} = c^2 m_{j,j+1} \int_{\Gamma^e_E} \partial_y N^{(j)}_a \partial_y N^{(j+1)}_b \, d\Gamma. \tag{180}$$

The global arrays appearing in (175) and (176) are obtained by the usual FE assembly process from the element arrays (179) and (180).

If all the shape functions $N^{(j)}_a$ are chosen to be identical, i.e., $N^{(j)}_a = \overline{N}_a$, and if in addition these boundary shape functions coincide with the trace of the domain shape functions N_a, then it is clear that many element matrices in (179) and (180) become identical up to a scaling factor. In fact, the matrices $\boldsymbol{C}^e$, $\boldsymbol{G}^e$, $\boldsymbol{A}^e$, $\boldsymbol{B}^e$ and $\boldsymbol{D}^e$ all become factors of the "boundary mass matrix" $\int_{\Gamma^e_E} \overline{N}_a \overline{N}_b \, d\Gamma$, and the matrices $\boldsymbol{E}^e$, $\boldsymbol{H}^e$ and $\boldsymbol{I}^e$ all become factors of the "boundary stiffness matrix" $\int_{\Gamma^e_E} \partial_y \overline{N}_a \partial_y \overline{N}_b \, d\Gamma$. Thus the calculation of the boundary element arrays becomes very efficient.

We choose to solve the semi-discrete system of equations consisting of (175)–(178) by standard Newmark time-stepping [42]. We solve all the interior equations *explicitly*, by diagonalizing the mass matrix via lumping, and all the equations associated with degrees of freedom on Γ_E *implicitly*. The reason for the implicit treatment of the time-integration on Γ_E is the lack of 'safe' mass lumping procedures for the non-standard ABC equations. More details on the time-integration scheme and other computational aspects can be found in [23].

7. Numerical experiments

In this section we present the results of some numerical experiments for the anisotropic wave Eq. (7), based on the FE formulation described in the previous section. As shown, this C^0 'primary-type' FE formulation allows the use of standard FEs with explicit time-integration in the interior of Ω. Unfortunately, such formulation is not possible with the convective wave equation, due to the appearance of mixed space–time derivatives. The latter would give rise to a nonsymmetric non-diagonal matrix $\boldsymbol{C}$ in the semi-discrete system of ODEs (175). In the absence of known 'safe' lumping techniques for nonsymmetric matrices, this would force us to make the whole scheme implicit in time. One-way to circumvent this difficulty is to use special FE spaces and mixed formulations; see., e.g., [43,44]. Resolving this matter is beyond the scope of this paper, and hence our numerical examples will not include convection and will concentrate on anisotropy.

The anisotropic problem can be illustrated using the setup shown in Fig. 5. The medium can be thought of as composed of unidirectional "fibers" embedded in an "environment" which is much softer, and thus possesses a much smaller wave speed. Two angles, measured with respect to the x direction (i.e., the direction normal to the boundary Γ_E) play a role in this scenario: the wave incidence angle θ and the angle of inclination of the fibers, θ_f. The two principal values of the tensor $\boldsymbol{\alpha}$ are α_{P_1} and α_{P_2}, which are the squares of the wave-speeds in the fiber direction and normal to the fibers. The components of α in the (x, y) directions can be obtained from the principal values via the transformation

$$\begin{Bmatrix} \alpha_{11} \\ \alpha_{12} \\ \alpha_{12} \end{Bmatrix} = \begin{bmatrix} \cos^2\theta_f & \sin^2\theta_f \\ \sin^2\theta_f & \cos^2\theta_f \\ -\sin\theta_f\cos\theta_f & \sin\theta_f\cos\theta_f \end{bmatrix} \begin{Bmatrix} \alpha_{P_1} \\ \alpha_{P_2} \end{Bmatrix}. \tag{181}$$

Naturally we will be interested to see how the high-order ABC performs in the presence of a mismatch between the directions of phase and group velocities. We recall that in the anisotropic case the condition for such mismatch is given by (21). For a symmetric configuration of geometry and initial conditions, this mismatch condition becomes

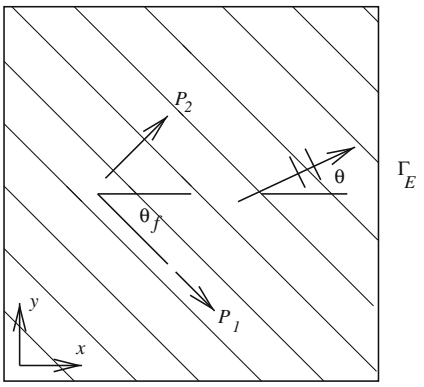

Fig. 5. Setup for the anisotropic case; the angle of incidence θ and the angle of inclination of the "fibers" θ_f.

$$\alpha_{12} \neq 0 \quad \text{and} \quad |\theta| > \theta_{cr} \equiv \tan^{-1}\left(\frac{\alpha_{11}}{|\alpha_{12}|}\right). \tag{182}$$

We can ask the following question: Given α_{P_1} and α_{P_2}, what is the fiber angle $\bar{\theta}_f$ that gives a minimal θ_{cr}? This would be the fiber angle associated with the largest phase-group mismatch possible, and would thus represent the "worst-case scenario." By substituting in (182) α_{11} and α_{12} given by (181), differentiating the result with respect to θ_f and equating to zero, we finally find

$$\cos\bar{\theta}_f = \sqrt{\frac{\alpha_{P_2}}{\alpha_{P_1}+\alpha_{P_2}}}. \tag{183}$$

Our numerical experiments are based on the following model. The computational domain Ω is a rectangle of dimensions 5×38. The support of the initial conditions is a rectangle of dimensions 4×3, whose left edge is located at the center of the west boundary Γ_W. Thus, at time $t = 0$ the initial wave is at a distance 1 from Γ_E and at a distance 17.5 from the south and north boundaries Γ_S and Γ_N. The domain is designed such that waves do not reach Γ_S and Γ_N at all during the simulation; this allows us to concentrate on the performance of the ABC on Γ_E. In the initial support, the function $\partial_t u(x, y, 0)$ is zero, while the function $u(x, y, 0)$ is a cross product of unit Hermite cubic "bells" in the x and y directions. (A unit Hermite cubic bell is a function that rises cubically from zero at point A to 1 at point B and then falls cubically back to zero at point C, while maintaining zero slopes at points A, B and C.) On Γ_W we prescribe the boundary condition $u = 0$ which is consistent with the initial condition.

We take a mesh of $50 \times 380 = 19,000$ square finite elements. We use bilinear shape functions for u and linear shape functions for all the ϕ_j on Γ_E. Thus we have 381 nodal points on the boundary Γ_E. We take a time-step size of $\Delta t = 0.025$. On Γ_E we use the high-order ABC with various orders P and with $a_j = 1$ for all $j = 1, \ldots, P$.

To calculate errors, we shall need a reference solution. To this end we calculate the solution in a reference domain Ω_{ref} which is much longer in the x direction than the original domain, i.e., of size 13.5×38. We use $135 \times 380 = 51,300$ elements in the reference model. The element size and time-step size are the same as in the original discretization. By doing 600 time-steps per simulation we ensure that waves reflected from the far east boundary of Ω_{ref} will not pollute the solution on Γ_E and in Ω. (We comment that in another set of experiments we let our code run for a much longer time, and verified that the scheme was stable.) We calculate the error e by

$$e(\boldsymbol{x}, t) = u^{hP}(\boldsymbol{x}, t) - u^h_{ref}(\boldsymbol{x}, t), \quad \boldsymbol{x} \in \Omega, \tag{184}$$

where u^{hP} is the computed solution in Ω using an ABC of order P, and u^h_{ref} is the restriction of the reference solution in Ω.

We relate here to the results of 11 simulations, whose parameters are summarized in Table 1. The ratio between the two principal wave speed squares in the simulation sets 0.#, 1.# and 2.# is 1 (isotropic), 2 and 10, respectively. Run 0.1 represents the isotropic case, Runs 1.1 and 2.1 represent the orthotropic case in which $\alpha_{12} = 0$, and in all the other runs $\alpha_{12} \neq 0$.

We remark that although in the orthotropic case simple coordinate transformation can reduce the wave equation to the isotropic one, we relate to this case here for two reasons. First, owing to the fact that in the orthotropic case there is never phase-group velocity mismatch (see (21)), this allows us to compare the various cases and check to what extent the accuracy is sensitive to the presence of such mismatch. Second, in thinking about future extension of the scheme to inhomogeneous media, it may well be the case that the medium will be orthotropic in one region and non-orthotropic in another.

Fig. 6 shows the variation in time of five sample reference solutions at point $x = 5, y = 19$ located at the center of the boundary Γ_E. Fig. 7 shows the errors, obtained with ABC order $P = 4$, corresponding to these five solutions. The strongly anisotropic case of Runs 2.1 and 2.3 where $\alpha_{P_2}/\alpha_{P_1} = 10$ generates a higher error than the isotropic case and the case $\alpha_{P_1}/\alpha_{P_2} = 2$. The deviation from orthotropy (namely the value of α_{12}) seems to affect the error at this location much less.

Table 1
Parameters of simulations of wave propagation in an anisotropic medium.

Run	α_{P_1}	α_{P_2}	$\bar{\theta}_f$	θ_f	α_{11}	α_{22}	α_{12}	θ_{cr}
0.1	1	1	–	–	1	1	0	–
1.1	1	0.5	54.7°	0°	1	0.5	0	–
1.2				30°	0.875	0.625	−0.217	76.1°
1.3				60°	0.625	0.875	−0.217	70.9°
2.1	0.1	1	17.5°	0°	0.1	1	0	–
2.2				15°	0.160	0.940	0.225	35.5°
2.3				20°	0.205	0.895	0.289	35.4°
2.4				40°	0.472	0.628	0.443	46.8°
2.5				60°	0.775	0.325	0.390	63.3°
2.6				75°	0.940	0.160	0.225	76.5°
2.7				90°	1	0.1	0	–

Fig. 8 demonstrates what happens if the artificial boundary is treated while paying no attention to the anisotropy (and thus to the entire issue of phase-group velocity mismatch). Thus, we use (112) with $\alpha_{12} = 0$ and $c_e^2 = \alpha_{11}$ instead of the definition (113). The errors for Run 2.3 are shown using the correct anisotropic ABC and an 'isotropic' ABC, with $P = 4$. It is clear that the errors generated by the latter are an order of magnitude larger. Moreover, it can be shown that if we take this incorrect ABC with $a_j < |\alpha_{12}|/\sqrt{\alpha_{11}\alpha_{22} - \alpha_{12}^2}$ then the problem becomes only *weakly* well-posed. This is in contrast to the strong well-posedness that Theorem 5.1 promises us for the correct ABC. (This weak instability of the incorrect ABC was not observed numerically, however. Only poor accuracy, as in Fig. 8, was observed in practice.)

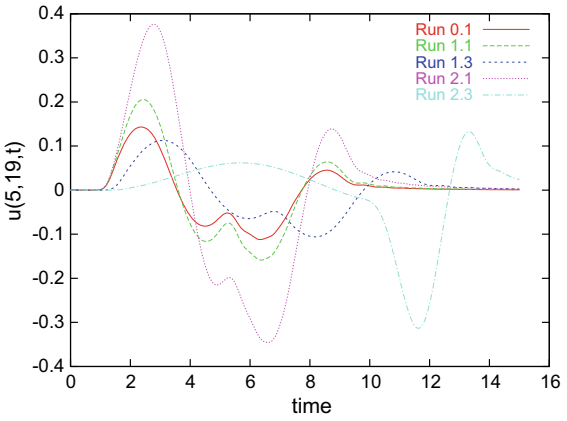

Fig. 6. Reference solutions at point $x = 5, y = 19$ (middle of Γ_E) as a function of time, for various sets of parameters, as described in Table 1.

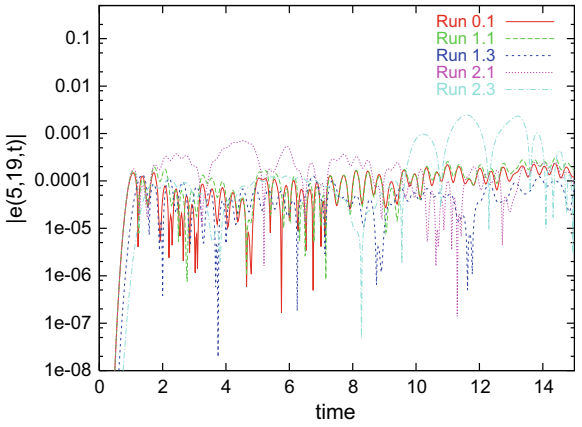

Fig. 7. Errors obtained with ABC order $P = 4$, at point $x = 5, y = 19$ (middle of Γ_E) as a function of time, corresponding to the solutions shown in Fig. 6.

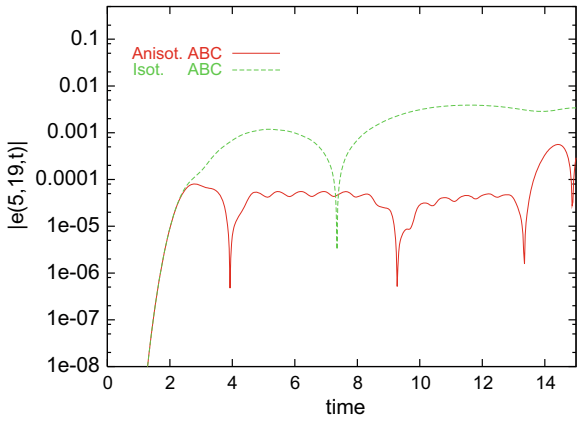

Fig. 8. Errors generated for Run 2.3 at point $x = 5, y = 19$ (middle of Γ_E) as a function of time, using the correct ABC and an 'isotropic' ABC. In both cases, $P = 4$.

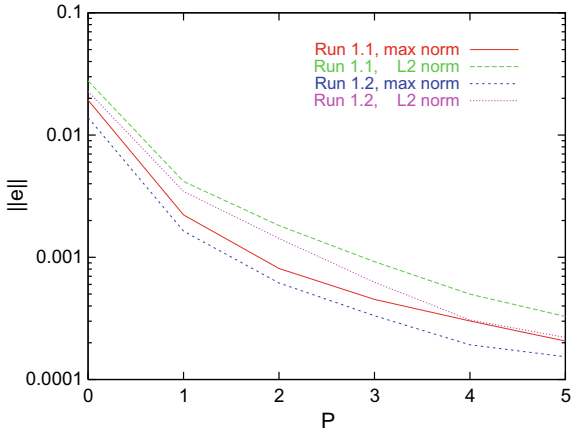

Fig. 9. Error (in the L_2 and max norm) as a function of the ABC order P for Runs 1.1 and 1.2, as described in Table 1.

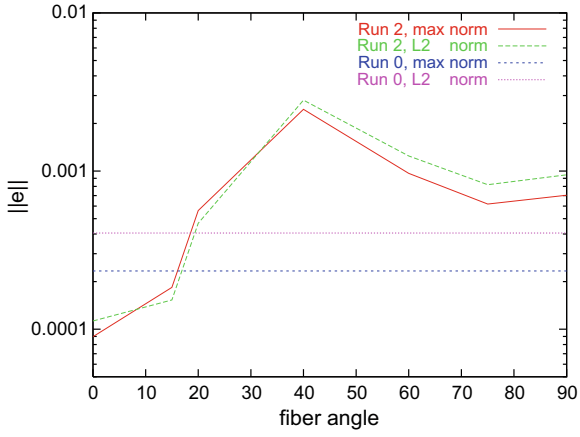

Fig. 10. Error (in the L_2 and max norm) for ABC order $P = 4$ as a function of the fiber angle θ_f for Runs 2.#, as described in Table 1.

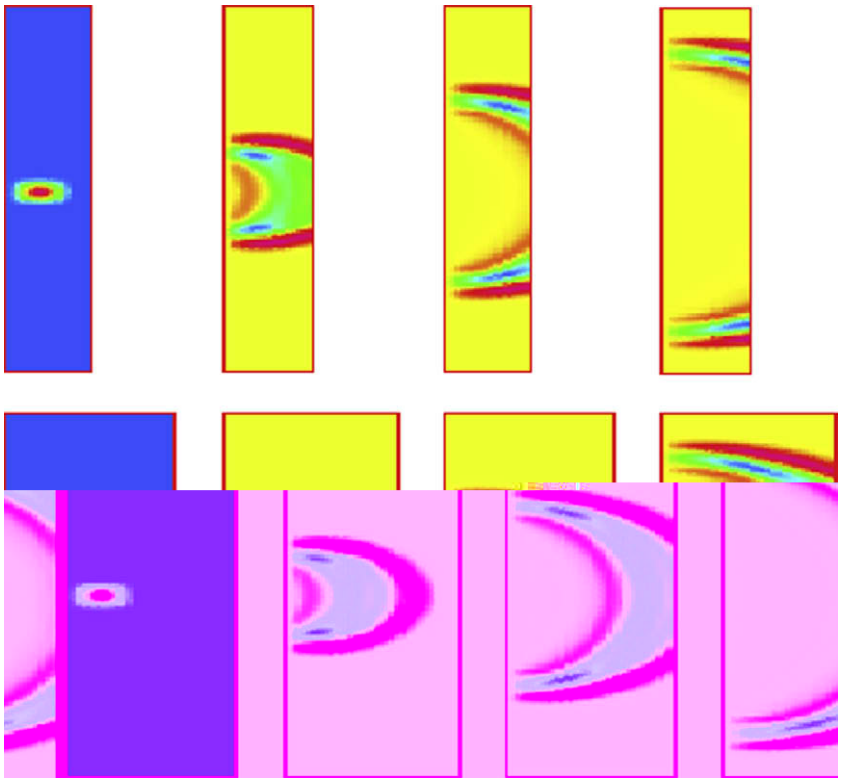

Fig. 11. Snapshots of computed solution with $P = 4$ (top) and reference solution (bottom) at times (from left to right) $t = 0, t = 5, t = 10$ and $t = 15$, for Run 0.1 (isotropic) as described in Table 1.

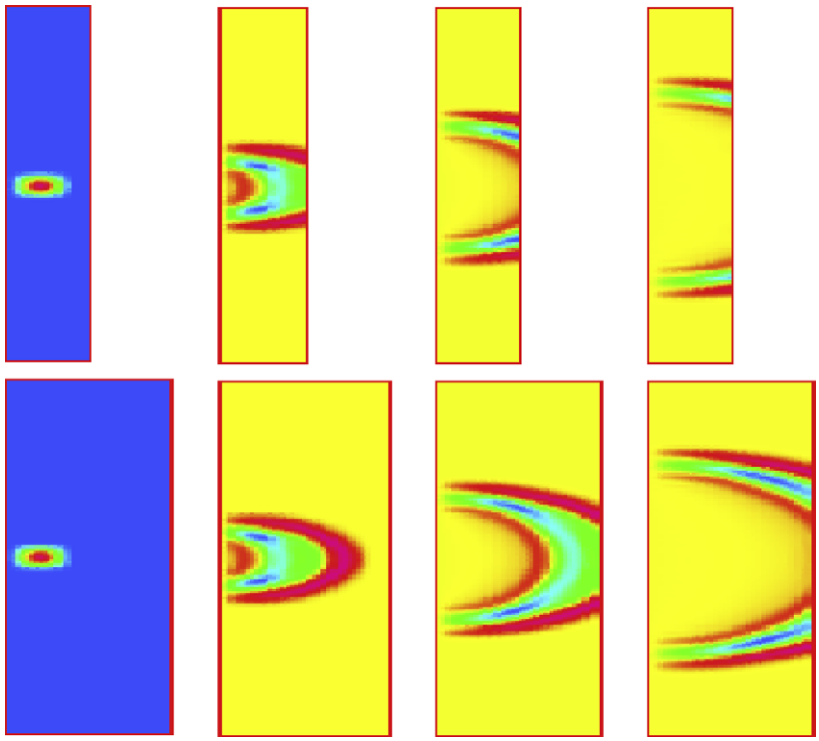

Fig. 12. Same as Fig. 11, but for Run 1.1 (orthotropic) as described in Table 1.

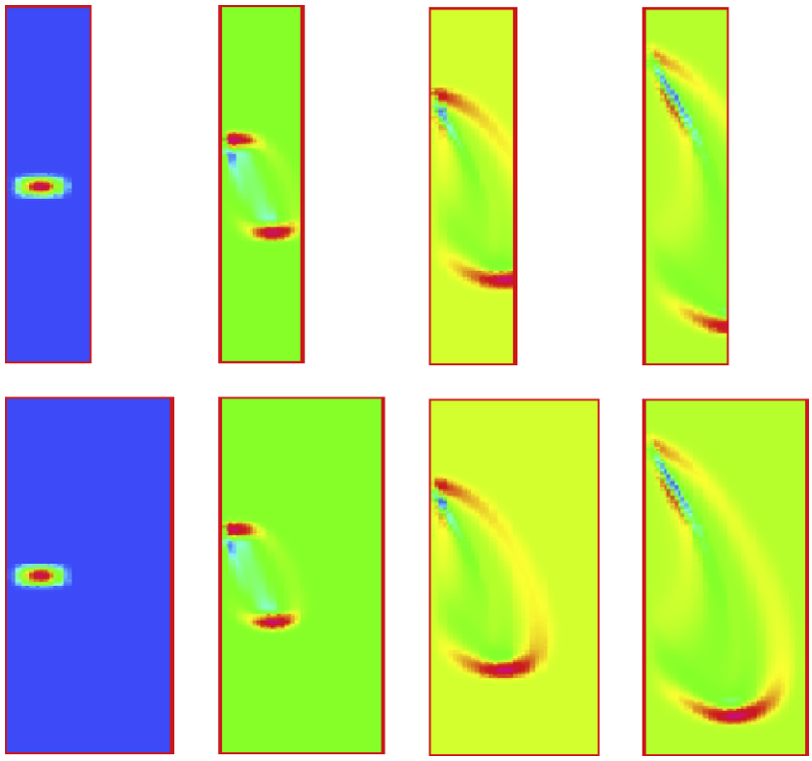

Fig. 13. Same as Fig. 11, but for Run 2.2 (strongly anisotropic) as described in Table 1.

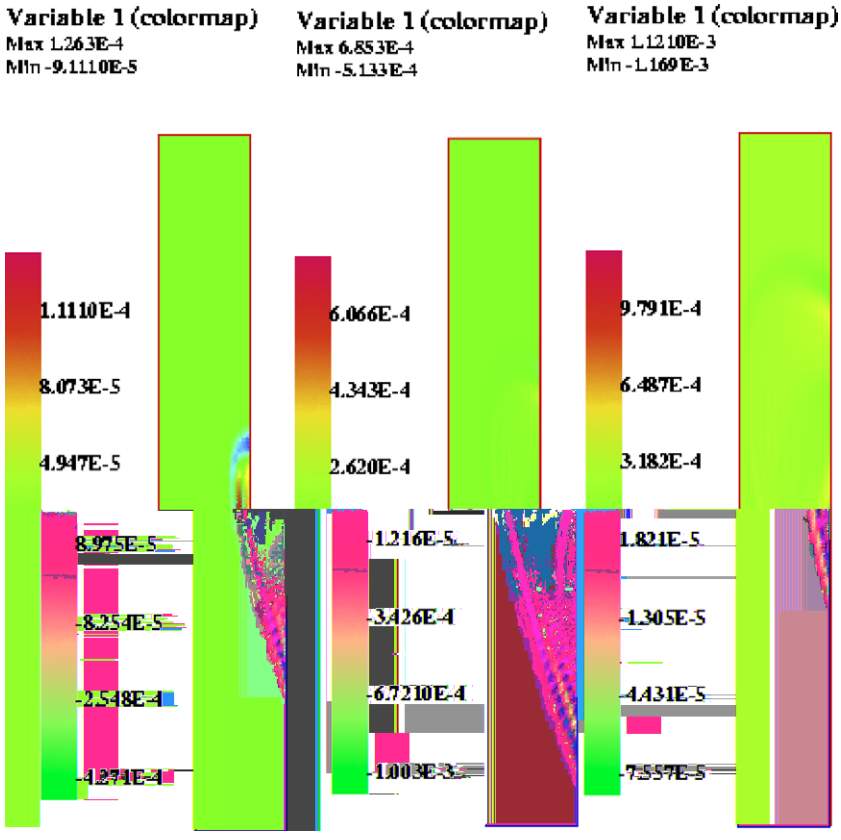

Fig. 14. Snapshots of errors corresponding to the solutions shown in Fig. 13 (i.e., for Run 2.2, with $P = 4$), at times $t = 5, t = 10$ and $t = 15$.

Fig. 9 shows the error in two norms – the L_2 norm and the maximum norm – as a function of the ABC error P, for Run 1.1 and Run 1.2 as described in Table 1. Initially the error decreases fast with P, but beyond $P = 4$ decreasing P further does not improve the accuracy significantly. The reason is that beyond $P = 4$ the time and space discretization errors start to dominate; see discussion in [23]. In fact, the use of *low-order* finite elements and time-integration schemes puts severe limits on the effective range of orders P. This difficulty may be alleviated by using high-order spectral elements and time discretization, as was demonstrated in [45], or high-order finite difference discretization in space and time, as was done in [19] and [23].

With $P = 4$, Fig. 10 shows the error norm values as a function of the Runs 2.#, whose parameters are described in Table 1. There is a significant variation in the error for the different fiber angles. However, it is interesting to note that this variation is not correlated with the mismatch of phase-speed and group-speed directions; see (182) and (183). In particular, since for Runs 2.# we have $\bar{\theta}_f = 17.5^\circ$ (see Table 1), one would perhaps expect that Run 2.2 with $\theta_f = 15^\circ$ and Run 2.3 with $\theta_f = 20^\circ$ would generate the largest errors, which is clearly not the case according to Fig. 10. Thus, as expected, the high-order ABC described in this paper is not sensitive to phase-group mismatch.

Figs. 10–12 show snapshots of computed solution with $P = 4$ (top) and reference solution (bottom) at times $t = 0, t = 5, t = 10$ and $t = 15$, for Run 0.1 (isotropic; Fig. 11), Run 1.1 (orthotropic; Fig. 12) and Run 2.2 (strongly anisotropic; Fig. 13). The reference solutions are shown in a domain which is twice as long as the computational domain Ω, but is shorter than the actual reference domain used. In all cases the agreement between the reference and computed solutions is excellent and no spurious reflection is observed.

Fig. 14 shows snapshots of the error e corresponding to the solutions shown in Fig. 13 (i.e., for Run 2.2, with $P = 4$). The error is globally small, and is distributed so that the largest error values are obtained where the wave front crosses the artificial boundary, as can be observed in Fig. 13.

8. Concluding remarks

In this paper we have shown how to extend the high-order Hagstrom–Warburton Absorbing Boundary Condition (ABC) to a general scalar linear time-dependent wave equation which represents acoustic wave propagation in anisotropic, convective and dispersive media. We proved that the reflection coefficient of this ABC decreases exponentially fast with increasing order P of the ABC, and that the problem is strongly well-posed. We also showed how to incorporate this ABC, for the anisotropic wave equation, in a finite element formulation. Of course, the high-order ABC can be used with other discretization methods as well.

Topics that were not dealt with here and are worth attention are, among others, developing corner conditions for the auxiliary variables (as was done for the isotropic case in [6] and [22]), accounting for various physical boundary conditions at the edges of the artificial boundary (in a wave-guide or half-space configuration), the extension to three dimensions, and fully developing and testing conditions which account for evanescent waves (as we have done in [20,23] for the isotropic wave equation). Concerning the last point, we remark that the ABC as tested, despite the fact that it is designed for propagating waves, is still guaranteed to be stable, and it will converge. Of course, the convergence for long time simulations might be rather slow, and it would be much more effective, as was shown in [20,23] to include the evanescent mode corrections.

Another important extension would be to inhomogeneous media, namely wave equations with varying coefficients. This was done in the isotropic stationary case in [23], for both layered and continuously stratified media. We believe that a similar extension can also be applied to the general wave equation considered here.

The importance of the present investigation, among other reasons, stems from its being a first step is extending our formulation to *anisotropic elasticity*, which is an important application in solid-earth geophysics. The elastic case is much more complicated than the scalar case. This is not merely a technical matter; in [31] Bécache et al. show for the PML that there are stability difficulties in the elastic case which do not exist in the acoustic case. We believe that similar difficulties apply to ABC methods. Due to the difference in methodologies, it is certainly worthwhile to study the extension of the high-order ABCs to elastodynamics.

Acknowledgments

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Appendix A. Proof that the auxiliary variables satisfy the wave equation

We consider the general wave Eq. (1) for u. Given the recursive relations (58) and (59) and the initial conditions (48) for the auxiliary variables ϕ_j, we wish to prove that each of the ϕ_j satisfies the same wave Eq. (1) that u satisfies, i.e.,

$$\mathcal{L}\phi_j \equiv (\alpha_{11}\partial_x^2 + 2\alpha_{12}\partial_{xy} + \alpha_{22}\partial_y^2 - \partial_t^2 - 2\beta_1\partial_{tx} - 2\beta_2\partial_{ty} - \nu)\phi_j = 0 \quad \text{in } D_E, \tag{A.1}$$

for $j = 1, \ldots, P + 1$. We prove this by induction.

First we prove that $\mathcal{L}\phi_1 = 0$. To this end, we apply the wave operator $\mathcal{L}$ to both sides of (58) to obtain

$$\mathcal{L}\mathcal{B}_0^+ u = \mathcal{L}\mathcal{B}_0^* \phi_1 \quad \text{in } D_E.$$

Since $\mathcal{L}u = 0$ and since L commutes with each of $\mathcal{B}_0^+$ and $\mathcal{B}_0^*$, we have

$$0 = \mathcal{B}_0^+ \mathcal{L}u = \mathcal{L}\mathcal{B}_0^+ u = \mathcal{L}\mathcal{B}_0^* \phi_1 = \mathcal{B}_0^* \mathcal{L}\phi_1.$$

Denoting $\psi_1 = \mathcal{L}\phi_1$ we thus have $\mathcal{B}_0^* \psi_1 = 0$. From the definition (64) of $\mathcal{B}_0^*$ this means

$$(\eta_0^* \partial_y + \tau_0^* \partial_t)\psi_1 = 0 \quad \text{in } D_E.$$

This is a one-way wave equation in the y direction. Its solution has the form

$$\psi_1(x,y,t) = f(x, y - (\eta_0^*/\tau_0^*)t), \tag{A.2}$$

for a general function f. (Note that $\tau_0^* > 0$ from (70).) From (48), $\psi_1(x,y,0) \equiv 0$, and we deduce from (A.2) that $f(x,y) \equiv 0$ in D_E. Therefore, $\psi_1 \equiv 0$ which means $\mathcal{L}\phi_1 \equiv 0$ in D_E.

Now we assume that $\mathcal{L}\phi_j = 0$ in D_E and we wish to prove that $\mathcal{L}\phi_{j+1} = 0$ in D_E. From (59) we have

$$0 = \mathcal{B}_j^+ \mathcal{L}\phi_j = \mathcal{L}\mathcal{B}_j^+ \phi_j = \mathcal{L}\mathcal{B}_j^- \phi_{j+1} = \mathcal{B}_j^- \mathcal{L}\phi_{j+1}.$$

Denoting $\psi_{j+1} = \mathcal{L}\phi_{j+1}$ we thus have $\mathcal{B}_j^- \psi_{j+1} = 0$. From the definition (63) of $\mathcal{B}_j^-$ this means

$$(-\xi\partial_x + \eta_j^- \partial_y + \tau_j^- \partial_t)\psi_{j+1} = 0 \quad \text{in } D_E. \tag{A.3}$$

Denoting $C_x = \xi/\tau_j^- > 0$ and $C_y = \eta_j^-/\tau_j^-$, (A.3) can be written as

$$(-C_x\partial_x + C_y\partial_y + \partial_t)\ \psi_{j+1} = 0 \quad \text{in } D_E. \tag{A.4}$$

This is a one-way wave equation in the direction $(-C_x, C_y)$, which is a vector pointing out of D_E towards Γ_E (incoming characteristic with respect to Ω); see Fig. 4. Eq. (A.4) can be written as

$$(-C_s\partial_s + \partial_t)\psi_{j+1} = 0 \quad \text{in } D_E,$$

where $C_s = \sqrt{C_x^2 + C_y^2}$ and $s \geqslant 0$ is the coordinate pointing away from Γ_E into D_E in the outgoing direction $-(-C_x, C_y)$. The solution of this equation is in the form

$$\psi_{j+1}(r,s,t) = f(r, s + C_s t), \tag{A.5}$$

for a general function f, where r the coordinate normal to s. From (48), $\psi_{j+1}(x,y,0) \equiv 0$, and we deduce from (A.5) that $f(r,s) \equiv 0$ in D_E for all $s \geqslant 0$. Therefore $\psi_{j+1} \equiv 0$ which means $\mathcal{L}\phi_{j+1} \equiv 0$ in D_E.

This completes the proof.

Appendix B. Derivation of boundary expression for the energy rate

We start from the definition of the energy E_0 in (158), and calculate the time rate of this energy:

$$\begin{aligned}\frac{dE_0}{dt} &= \int_\Omega \left[(\partial_t u)(\partial_t^2 u) + \boldsymbol{\alpha}\nabla u \cdot \partial_t \nabla u + v u \partial_t u\right] d\Omega = \int_\Omega \left[\partial_t u(\nabla \cdot \boldsymbol{\alpha}\nabla u - 2\boldsymbol{\beta}\cdot\partial_t\nabla u - v u) + \boldsymbol{\alpha}\nabla u \cdot \partial_t \nabla u + v u \partial_t u\right] d\Omega \\ &= \int_\Gamma (\partial_t u)\boldsymbol{\alpha}\nabla u \cdot \mathbf{n}\, d\Gamma - \int_\Omega \boldsymbol{\beta}\cdot\nabla((\partial_t u)^2)\, d\Omega = \int_\Gamma [\boldsymbol{\alpha}\nabla u \cdot \mathbf{n} - \boldsymbol{\beta}\cdot\mathbf{n}\partial_t u]\partial_t u\, d\Gamma.\end{aligned}$$

The first equality is obtained by differentiating with time the integrand in (158), the second equality is obtained by using the wave Eq. (1) to express $\partial_t^2 u$, the third equality is obtained by "integration by parts" (namely applying Green's identity) to the first term with $\boldsymbol{\alpha}$, and the last equality is obtained by applying the divergence theorem to the term with $\boldsymbol{\beta}$. This calculation proves (161), which is what we wanted to show.

Now we consider the energy E_1 in (159). We denote $2e$ the integrand in (159) and thus

$$E_1 = \int_\Omega e\, d\Omega. \tag{B.1}$$

We note that the wave Eq. (1) can we written as

$$\mathcal{L}u \equiv \nabla \cdot \boldsymbol{\kappa}\nabla u - (D_t^2 u + v u) = 0, \tag{B.2}$$

where D_t is defined by (160). Then we calculate:

$$\begin{aligned}\frac{dE_1}{dt} &= \int_\Omega \partial_t e\, d\Omega = \int_\Omega (D_t e - \boldsymbol{\beta}\cdot\nabla e)\, d\Omega = \int_\Omega D_t e\, d\Omega - \int_\Gamma e\boldsymbol{\beta}\cdot\mathbf{n}\, d\Gamma = \int_\Omega \frac{1}{2} D_t\left[(D_t u)^2 + \boldsymbol{\kappa}\nabla u\cdot\nabla u + \nu u^2\right] d\Omega - \int_\Gamma e\boldsymbol{\beta}\cdot\mathbf{n}\, d\Gamma \\ &= \int_\Omega [D_t u D_t^2 u + \boldsymbol{\kappa}\nabla D_t u\cdot\nabla u + \nu u D_t u]\, d\Omega - \int_\Gamma e\boldsymbol{\beta}\cdot\mathbf{n}\, d\Gamma = \int_\Omega [\nabla\cdot\boldsymbol{\kappa}\nabla u D_t u + \boldsymbol{\kappa}\nabla u\cdot\nabla D_t u]\, d\Omega - \int_\Gamma e\boldsymbol{\beta}\cdot\mathbf{n}\, d\Gamma \\ &= \int_\Gamma \boldsymbol{\kappa}\nabla u\cdot\mathbf{n} D_t u\, d\Gamma - \int_\Gamma e\boldsymbol{\beta}\cdot\mathbf{n}\, d\Gamma = \int_\Gamma \left[\boldsymbol{\kappa}\nabla u\cdot\mathbf{n} D_t u - \frac{\boldsymbol{\beta}\cdot\mathbf{n}}{2}((D_t u)^2 + \boldsymbol{\kappa}\nabla u\cdot\nabla u + \nu u^2)\right] d\Gamma.\end{aligned}$$

The eight equalities above are obtained, respectively, from (i) Eq. (B.1), (ii) the definition of D_t, (iii) the divergence theorem, (iv) the definition of e, (v) applying the D_t derivative, (vi) Eq. (B.2), (vii) the divergence theorem, and (viii) the definition of e. The end result is exactly (162), which is what we wanted to show.

References

[1] T. Hagstrom, Radiation boundary conditions for the numerical simulation of waves, Acta Numer. 8 (1999) 47–106.
[2] D. Givoli, Computational absorbing boundaries, in: S. Marburg, B. Nolte (Eds.), Computational Acoustics of Noise Propagation in Fluids, Springer, Berlin, 2008, pp. 145–166 (Chapter 5).
[3] J.P. Bérenger, A perfectly matched layer for the absorption of electromagnetic waves, J. Comput. Phys. 114 (1994) 185–200.
[4] F. Collino, High order absorbing boundary conditions for wave propagation models. straight line boundary and corner cases, in: R. Kleinman et al. (Eds.), Proc. 2nd Int. Conf. on Mathematical and Numerical Aspects of Wave Propagation, SIAM, Delaware, 1993, pp. 161–171.
[5] S. Asvadurov, V. Druskin, M. Guddati, L. Knizherman, On optimal finite difference approximation of PML, SIAM J. Numer. Anal. 41 (2003) 287–305.
[6] T. Hagstrom, T. Warburton, A new auxiliary variable formulation of high-order local radiation boundary conditions: corner compatibility conditions and extensions to first order systems, Wave Motion 39 (2004) 327–338.
[7] D. Givoli, Non-reflecting boundary conditions: a review, J. Comput. Phys. 94 (1991) 1–29.
[8] B. Engquist, A. Majda, Radiation boundary conditions for acoustic and elastic wave calculations, Commun. Pure Appl. Math. 32 (1979) 313–357.
[9] A. Bayliss, E. Turkel, Radiation boundary conditions for wave-like equations, Commun. Pure Appl. Math. 33 (1980) 707–725.
[10] R.L. Higdon, Numerical absorbing boundary conditions for the wave equation, Math. Comput. 49 (1987) 65–90.
[11] R.L. Higdon, Radiation boundary conditions for dispersive waves, SIAM J. Numer. Anal. 31 (1994) 64–100.
[12] M.J. Grote, J.B. Keller, Exact nonreflecting boundary conditions for the time dependent wave equation, SIAM J. Appl. Math. 55 (1995) 280–297.
[13] M.J. Grote, J.B. Keller, Nonreflecting boundary conditions for time dependent scattering, J. Comput. Phys. 127 (1996) 52–65.
[14] T. Hagstrom, S.I. Hariharan, A formulation of asymptotic and exact boundary conditions using local operators, Appl. Numer. Math. 27 (1998) 403–416.
[15] M.N. Guddati, J.L. Tassoulas, Continued-fraction absorbing boundary conditions for the wave equation, J. Comput. Acoust. 8 (2000) 139–156.
[16] D. Givoli, B. Neta, High-order non-reflecting boundary scheme for time-dependent waves, J. Comput. Phys. 186 (2003) 24–46.
[17] D. Givoli, B. Neta, I. Patlashenko, Finite element solution of exterior time-dependent wave problems with high-order boundary treatment, Int. J. Numer. Meth. Eng. 58 (2003) 1955–1983.
[18] D. Givoli, High-order local non-reflecting boundary conditions: a review, Wave Motion 39 (2004) 319–326.
[19] T. Hagstrom, M. de Castro, D. Givoli, D. Tsemach, Local high order absorbing boundary conditions for time-dependent waves in guides, J. Comput. Acoust. 15 (2007) 1–22.
[20] T. Hagstrom, T. Warburton, D. Givoli, Radiation boundary conditions for time-dependent waves based on complete plane wave expansions, J. Comput. Appl. Math., in press.
[21] J. Diaz, P. Joly, An analysis of higher order boundary conditions for the wave equation, SIAM J. Appl. Math. 65 (2005) 1547–1575.
[22] D. Givoli, T. Hagstrom, I. Patlashenko, Finite element formulation with high order absorbing boundary conditions for time-dependent waves, Comput. Meth. Appl. Mech. Eng. 195 (2006) 3666–3690.
[23] T. Hagstrom, A. Mar-Or, D. Givoli, High-order local absorbing conditions for the wave equation: extensions and improvements, J. Comput. Phys. 227 (2008) 3322–3357.
[24] T. Hagstrom, T. Warburton, Complete radiation boundary conditions: minimizing the long time error growth of local methods, submitted for publication.
[25] T. Hagstrom, S.I. Hariharan, D. Thompson, High-order radiation boundary conditions the convective wave equation in exterior domains, SIAM J. Sci. Comput. 25 (2003) 1088–1101.
[26] M.E. Goldstein, Aeroacoustics, McGraw-Hill, New York, 1976.
[27] D.H. Rudy, J.C. Strikwerda, A nonreflecting outflow boundary condition for subsonic Navier–Stokes calculations, J. Comput. Phys. 36 (1980) 55–70.
[28] A. Bayliss, E. Turkel, Far field boundary conditions for compressible flows, J. Comput. Phys. 48 (1982) 182–199.
[29] A. Bayliss, E. Turkel, Outflow boundary conditions for fluid dynamics, SIAM J. Sci. Stat. Comput. 3 (1982) 250–259.
[30] K. Helbig, L. Thomsen, 75-plus Years of anisotropy in exploration and reservoir seismics: a historical review of concepts and methods, Geophysics 70 (2005) 9–23.
[31] E. Bécache, S. Fauqueux, P. Joly, Stability of perfectly matched layers, group velocities and anisotropic waves, J. Comput. Phys. 188 (2003) 399–433.
[32] D. Appelö, T. Hagstrom, G. Kreiss, Perfectly matched layers for hyperbolic systems: general formulation, well-posedness and stability, SIAM J. Appl. Math. 67 (2006) 1–23.
[33] K.C. Meza-Fajardo, A.S. Papageorgiou, A nonconvolutional, split-field, perfectly matched layer for wave propagation in isotropic and anisotropic elastic media: stability analysis, Bull. Seism. Soc. Am. 98 (2008) 1811–1836.
[34] N. Yarvin, V. Rokhlin, Generalized gaussian quadratures and singular value decompositions of integral operators, SIAM J. Sci. Comp. 20 (1998) 699–718.
[35] B. Gustafsson, H.-O. Kreiss, J. Oliger, Time Dependent Problems and Difference Methods, Wiley, NY, 1995.
[36] H.O. Kreiss, Initial boundary value problems for hyperbolic systems, Commun. Pure Appl. Math. 23 (1970) 277–298.
[37] R.L. Higdon, Initial boundary value problems for linear hyperbolic systems, SIAM Rev. 28 (1986) 177–217.
[38] R.L. Higdon, Radiation boundary conditions for elastic wave propagation, SIAM J. Numer. Anal. 27 (1990) 831–870.
[39] L.N. Trefethen, Group velocity implementation of the stability theory of Gustafsson, Kreiss and Sundström, J. Comput. Phys. 49 (1983) 199–217.
[40] A. Majda, S. Osher, Initial boundary value problems for hyperbolic equations with uniformly characteristic boundary, Commun. Pure Appl. Math. 28 (1975) 607–675.
[41] T. Ha-Duong, P. Joly, On the stability analysis of boundary conditions for the wave equation by energy methods. Part I: The homogeneous case, Math. Comput. 62 (1994) 539–563.
[42] T.J.R. Hughes, The Finite Element Method, Prentice-Hall, Englewood Cliffs, NJ, 1987.

[43] G. Gabard, R.J. Astley, M. Ben Tahar, Stability and accuracy of finite element methods for flow acoustics. II: Two-dimensional effects, Int. J. Numer. Meth. Eng. 63 (2005) 974–987.
[44] A.-S. Bonnet-Ben Dhia, È.-M. Duclairoir, G. Legendre, J.-F. Mercier, Time-harmonic acoustic propagation in the presence of a shear flow, J. Comput. Appl. Math 204 (2007) 428–439.
[45] L. Kucherov and D. Givoli, High Order Absorbing Boundary Conditions Incorporated in a Spectral Element Formulation, Commun. Numer. Meth. Eng., in press. doi:10.1002/cnm.1188.